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VOLUME THREE

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VOLUME III

HOUSTON, TEXAS
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THE INAUGURAL LECTURES

THE INTRODUCTION OF WESTERN LEARNING INTO JAPAN[1]

THE FIRST PERIOD

THE intercourse of Japan with the West began, in the middle of the sixteenth century, with the coming of the Portuguese ships to the coast of Kyûshû (1543). Not long after came the English, the Dutch, and the Spanish. The Portuguese and Spaniards were indiscriminately called *Nanban,* which means "southern foreigners," as their possessions in Asia lay to the south of Japan, just as in the present day we speak of all white people as *Seiyôjin,* "men of the western seas."

At this period the shogunate of the Ashikaga family was tottering toward its fall. The *Shôgun,* or *Sei-I-Tai-Shô-Gun* (which was the full title, meaning "Generalissimo for the Subjugation of Barbarians"), was the head of the military class and *de facto* ruler of the country; for the Emperor and the civil lords who formed his court had very little or no real power, although they were reverenced by the people and outwardly treated with honor and deference by the shôgun and his followers. The office of shôgun had at the time of the first coming of the Portuguese been hereditary in the Ashikaga family for over two hundred years, but in the feeble hands of its latest representatives its authority had gradually been weakened until the great military chiefs throughout the country paid but little attention to their orders and were continually fighting against one another in

[1] A lecture presented at the inauguration of the Rice Institute, by the Right Honorable Baron Dairoku Kikuchi, *Rigakuhakushi,* M.A., LL.D., Privy Councilor, President of the Imperial Academy, Honorary Professor of the Imperial University of Tokyo.

a struggle for self-aggrandisement. Among them appeared three great men: the first was Nobunaga (of the Ota family), who deposed the last of the Ashikaga shôguns (1573) and brought the whole of central Japan under his authority. After Nobunaga was killed by one of his own generals (1582), Hideyoshi, another of his generals, better known by his subsequent title of Taikô, extended his power over the whole country. After the death of the Taikô in 1598, Iyeyasu, the head of the Tokugawa family, who had been gradually strengthening himself, patiently biding his time under Nobunaga and Hideyoshi, became shôgun in 1603 and established his government in Yedo. Iyeyasu and his descendants held the shogunate for fifteen generations, and were the real rulers of the land for over two centuries and a half, during which period Japan enjoyed a most profound peace, and learning and the arts flourished under the patronage of shôguns and *daimyôs* (or feudal lords).

The above brief outline is necessary for a clear understanding of the environment in which the first introduction of Western learning took place. The Portuguese were welcomed by the military chiefs principally for the sake of firearms, which were first introduced by them, and which of course gave to those possessed of them an immense advantage over their enemies. Their use and making were eagerly acquired, and already in 1553 the shôgun Yoshiteru had guns made for him at Anato, in the province of Omi, not far from Kyoto. The introduction of firearms necessarily brought about a change in tactics and fortification, but it is uncertain how much the military chiefs learned in these things from the Portuguese.

Not very long after the first coming of the Portuguese, the Jesuit missionaries arrived. They also were well received by the military lords of Kyûshû, several of the most

powerful of whom became converts; so that Christianity at first made rapid progress, spreading not only in Kyûshû and adjoining provinces, but also in the neighborhood of Kyoto, and later even in northwestern Japan. The shôgun Yoshiteru, mentioned above, is himself said to have been among the converts. Nobunaga also was at first favorable and built for them a church in Kyoto called the *Nanbanji,* or "Temple of the Southern Foreigners"; but he afterward repented of this, and his successor Hideyoshi issued orders for the suppression of Christianity. It may be mentioned that the motives which influenced both Nobunaga and Hideyoshi were entirely political and not at all religious.

Iyeyasu, his successor in power, was friendly to foreigners, and among others treated a Dutchman named Jan Joost and an English pilot, Will Adams, who arrived in a Dutch ship in 1600, with great consideration; he was eager to learn from them about the world outside of Japan. He and his successors, however, looked with no favorable eyes upon missionaries or their converts, for they were a source of trouble everywhere on account of their intolerance and quarrelsome attitude toward those of other faiths. They were, moreover, suspected of political intrigue against the shogunate and against the country; so orders were issued expelling not only the missionaries but all Portuguese and Spaniards, and forbidding people to profess Christianity on pain of death or exile. This state of affairs culminated in the breaking out in 1637 of rebellion in Shimabara, near Nagasaki, whither had flocked not only Christians driven by persecution from other parts of the country, but also a large number of malcontent and turbulent spirits, followers of lords who had fought unsuccessfully against the Tokugawas. The rebellion was put down early in the next year, and most stringent measures were taken to stamp out Christianity al-

together. Already in 1630 an order had been issued by which all foreign books, without exception, were interdicted; for although it was primarily aimed at religious books, it was impossible to make such a distinction without a knowledge of European languages. In 1635 another order was issued prohibiting all traveling abroad under the penalty of death. Thus, about ninety years after the first arrival of the Portuguese ships, all foreign intercourse was forbidden except such as was permitted with the Dutch and the Chinese under severe restrictions.[1]

It is hard to say exactly how much learning had been transmitted by the Portuguese and Spaniards during this period. Among the missionaries were some skilled in medicine and surgery, and their method of treating wounds seems to have been especially appreciated; thus an elementary knowledge of "Nanban" surgery, as it was called, as well as of the warlike art of gunnery, seems to have been acquired by the Japanese from them. A man named Hayashi, who was put to death (1646) for professing Christianity, had acquired some knowledge of Western mathematics and astronomy, which he transmitted to his pupil Kobayashi; he had translated and published a work on astronomy (1635), which stands second in the list of the translations of Western books into Japanese, the first being Æsop's "Fables," translated and published early in the seventeenth century, although perhaps neither of these was a translation in the strict sense of the term, but rather a compilation. It is also interesting to note that some of the great military lords used seals bearing their names in Latin letters. There are several Japanese words of Portuguese and Spanish origin, which bear testimony to the introduction in those days of

[1] The English had previously abandoned the field, and their request to resume intercourse in 1673 was not entertained.

various manufactures; such, for example, as *biidoro* (glass, Portuguese *vidro*), *botan* (button, P. *botao*), *birôdo* (velvet, P. *veludo*), *kappa* (rain-cloak, Spanish *capa*), *meriyasu* (knit-work, S. *medias*), etc. On the whole, the amount of Western knowledge introduced during this period cannot have been very great.

THE SECOND PERIOD

Our intercourse with the western world after the exclusion of the Portuguese and Spaniards was through the Dutch, who were permitted to come to trade in the single port of Nagasaki; even here they were confined to a small quarter of the town known as Dejima, and the trade was subjected to rigorous restrictions and placed under the strict surveillance of officials of the shogunate. A corps of interpreters was maintained in Nagasaki, the office being hereditary in certain families, as was the case in those days with almost all professions; but even they were not permitted to read or possess any foreign books, so that their knowledge of the Dutch language was entirely oral; it was not till 1745 that this prohibition was removed. Once a year (afterward once every four years) the Dutch "capitan," or chief factor, was required to come to Yedo to pay his respects to the shôgun; and these visits played an important part in the introduction of Western knowledge into Japan, for scholars in Yedo took advantage of these occasions to "interview," usually with official sanction, the "capitan" and those who accompanied him, asking all sorts of questions on all sorts of subjects. It is pathetic in some cases to read of distinguished scholars, in their simplicity and zeal for knowledge, reverently asking questions such as the factors could scarcely have understood; yet as in those days communication between Nagasaki and

Yedo was not easy, and as the "capitans" were accompanied by physicians (rarely by such men as Kaempfer, Thünberg, and Von Siebold, who took advantage of their visits to see the interior of Japan), those interviews were really a great opportunity for those who were eager to learn about the West.

Although the first three shôguns of the Tokugawa family took such strong measures to suppress Christianity, even going so far as to cut off almost all foreign intercourse and to interdict all foreign books, yet both they and their successors were patrons of learning and the arts, and were by no means averse to the introduction of useful knowledge from the West. Several of the interpreters and others who had picked up some medical, or rather surgical, knowledge from the Dutch physicians in Nagasaki were appointed physicians to the shôgun, an example which was followed by the daimyôs. Arai Hakuseki[1] (1657–1725), a great Chinese scholar and a trusted adviser of the shôgun Iyenobu, sixth shôgun of the Tokugawa family (1709–1711), interviewed at the command of the shôgun a Franciscan priest who had arrived in 1709 at Osumi in Kyûshû and had been summoned to Yedo, where he was kept in confinement. This priest seems to have been a man of some attainments, and an account of the interviews and their results, supplemented by subsequent interviews with Dutch "capitans," was embodied in two books entitled *Sairan Igen* (1713) and *Seiyô Kibun* (1715). These books, written by a man of Arai's standing and scholarship, gave certain importance and prestige to their contents—*i.e.,* to matters Western—which they had not hitherto possessed, and thus opened the way for the introduction of Western learning. For this reason Arai Hakuseki is regarded as its pioneer.

[1] All the names of men are given in the usual Japanese way—*i.e.,* with the family name first.

The accession of the eighth shôgun, Yoshimune (1716–1745), gave a great impetus to the introduction of Western learning. He was specially interested in astronomy, and had a celestial globe and a sun-dial made for himself; he also sent to Nagasaki (1719) for Nishigawa Joken, who had obtained some knowledge of astronomy from Kobayashi (see above), and finally established an astronomical observatory in Yedo in 1744. Up to this time, foreign books being prohibited, the little Western knowledge that had been acquired had been either through oral communications or through Chinese translations, which had filtered through to Japan, Chinese books not coming within the category of prohibited books, for Chinese was the language of scholars in Japan to within very recent times, just as Latin was the language of the learned in Europe of the Middle Ages. But now Yoshimune removed this interdiction on foreign books, excepting those on religion (1720). In 1738 a book on astronomy presented to the shôgun by the Dutch challenged his admiration by the excellence of its illustrations, and seeking for some one to read the explanations of the plates, he ordered a man named Aoki Bunzô (1698–1769) to begin the study of the Dutch language. Aoki learned some Dutch words from the interpreters who came to Yedo with the factor, but not making much headway he went to Nagasaki, where incidentally he was instrumental in getting an order from the government allowing interpreters to read books. He returned to Yedo, having succeeded in learning only some five hundred words, which is very good evidence of the extreme difficulty of the task in those days. I regret that the space at my command does not allow me to enter into an explanation of the various obstacles that lay in the way of such study.

The death of Yoshimune in 1751—he had retired from

active life in 1745—was a blow to the advancement of Western learning; but the impetus given could not be checked. Thus the Observatory, although abolished in 1757, was re-established in 1765. Objects brought by the Dutch began to be sought for as curios and articles of virtu, books among the rest. About this time there also flourished an eccentric and versatile genius called Hiraga Gennai; among other evidences of his originality, he in 1770 constructed an electric machine like one which he had seen in Nagasaki.

THE THIRD PERIOD

But now comes an event of the first importance in the introduction of Western knowledge, namely, the translation and publication of the first work on anatomy in 1774, through the joint efforts of Maeno, Sugita, and others. Up to this time the only attempt made to read Dutch books had been made by Aoki, who, as already mentioned, succeeded with enormous difficulty in learning several hundred words; some knowledge of astronomy had been acquired through Chinese translations, and the Dutch medicine, so called, had been represented by an empirical practice of surgery.

Maeno Ryôtaku (1723–1803), a physician to the Lord of Nakatsu, was a man of great originality and perseverance, and Sugita Genpaku (1733–1817), a surgeon of the so-called Dutch school, was a man of kindred spirit. Indeed, most of those who were pioneers in the introduction of Western knowledge into Japan were men of original ideas and advanced views, eager and indefatigable in their pursuit of knowledge, often at the risk of personal inconvenience or danger. Maeno, impelled by a desire to read Dutch, but unable to get much assistance from the interpreters who came with the Dutch to Yedo, became a pupil of

Aoki, who taught him all he knew. Both he and Sugita derived much profit from a Dutch physician who came one year to Yedo. Not content with this, Maeno went to Nagasaki for several months in 1770, and returned with his vocabulary extended to some seven hundred words, and with a Dutch dictionary and a book on anatomy ("Tafel Anatomia"). The next year he and Sugita were present at the dissection of an executed criminal in Senju, a suburb of Yedo, where the executions generally took place. Such dissections began about this time to be occasionally made on the bodies of executed criminals, at the request of influential physicians, the knife being usually wielded by the executioner, a member of the low *Eta* caste (the only caste that existed in old Japan, and now entirely done away with), who pointed out to those present such organs as he happened to know. The fact that such dissections took place is an evidence of the universal spirit of intellectual unrest which distinguished this age, and of which indeed the desire for Western knowledge was one of the manifestations. Up to this time, however, doctors had not dared to question, openly at least, the truth of the old Chinese teaching about the constitution of the human body, but had been enveloped in doubt and perplexity. On that memorable day Maeno, Sugita, and a few others, comparing what they saw with the figures in the "Tafel Anatomia" that Maeno had brought from Nagasaki, and of which Sugita by a most happy coincidence had also secured a copy, were greatly impressed by their faithfulness to nature, and then and there they determined to devote their lives to exploring the new domain of knowledge thus opened to their view. The very next day they met at Maeno's house and began the work of deciphering the book —for it was deciphering, and nothing less. To this task Maeno brought his knowledge of some seven hundred words

and the dictionary, while some of them did not even know the alphabet; but, nothing daunted, they set to work and toiled for three whole years, until 1774, during which time the band was joined by some new members and deserted by some old ones. The names of the eight who were constant in their devotion to the self-appointed task deserve to be mentioned here, viz., Maeno Ryôtaku, Sugita Genpaku, Katsuragawa Hoshû (1751–1809), Nakagawa Junnan, Ishikawa Genjô, Toriyama Shôen, Mine Shuntai, and Kiriyama Seitetsu. Sugita always wrote out at night what had been deciphered during the day, making corrections and revisions as the work progressed, so that at the end of three years the translation was completed simultaneously with the deciphering. The publication of this work, entitled *Kaitai Shinsho,* or "New Anatomy," marks an epoch in the history of the introduction of Occidental civilization into Japan; for not only was it a great training and education to those who took part in it, giving them confidence and power, and making them, as it were, the center of the new movement, but it made known to a much wider circle than before the existence of an entirely new system of learning and roused a spirit of inquiry in bolder minds, many of whom joined the pioneers as associates and pupils and became their successors in carrying on the work.

Maeno was interested in the Dutch language, and wrote several books in order to make its study and translation easier, while Sugita devoted himself more especially to the advancement of the knowledge and practice of the new medicine. From this time on, the introduction of Western knowledge was placed on a firmer basis; for original books became accessible to those who took pains enough—great pains, no doubt, but not to be compared with those of Maeno and his fellows. To this result Otsuki Gentaku (1757–

1827), a pupil of Maeno and of Sugita, contributed very greatly, both by his personal teaching and by his books, among which may be specially mentioned one entitled *Rangaku Kaieti,* or "Introduction to the Study of Dutch" (1788). Many now came to him to get help in reading Dutch; one of his pupils, Inamura Sanpaku, compiled a Dutch-Japanese dictionary containing eighty thousand words, after a Dutch-French dictionary of François Halma, and type-printed thirty copies of it by subscription in 1796. An abridged edition containing thirty thousand words was afterward made by his pupil Fujibayashi, of which one hundred copies were printed in 1810. Another dictionary based on the same Dutch-French dictionary was compiled at Nagasaki by a Dutchman, Hendrik Doeff, a resident in Nagasaki for seventeen years, with the assistance of Yoshiwo Gonnosuke and other interpreters. This was completed in 1816, but was not printed until much later (1855–1858). It was afterward known as "Doeff Halma" to distinguish it from the "Yedo Halma" of Inamura. Various abridged dictionaries were compiled, and some of them printed, all tending to make the acquiring of the Dutch language easier; but those of Inamura and Doeff continued to be standard works, and as they were both out of print, they used to be copied by poor students, who thereby earned money and at the same time increased their knowledge of the Dutch vocabulary.

The so-called Dutch medicine had up to this period been confined, as already mentioned, to the practice of surgery, but Udagawa Genzui (1755–1797), a physician to the Lord of Tsuyama, seeing the errors of the old Chinese school of medicine, became a pupil and afterward an eminent member of the band of Dutch scholars, and at the suggestion of Katsuragawa (one of Maeno's co-workers) took up the study of a Dutch work on medicine by one Johannes Gorter. Al-

though he had the invaluable assistance not only of Katsuragawa, but also of Maeno, Sugita, Otsuki, and others, who all earnestly desired his success for the sake of the advancement of their cause, he had to contend not only with the difficulty of the subject-matter itself, but also with that of the language, as yet scarcely mastered. It took him nine years to complete the translation of the work, which was published in the tenth year (1793) under the name of *Naika Sen-yô,* or "Elements of Internal Medicine." This was the first time that the Western system of (internal) medicine was made known to the Japanese. Udagawa afterward wrote several other books on medicine. His adopted son, Udagawa Genshin (1769–1834), a pupil of Otsuki, was a very good Chinese scholar, and is said to have been a great help to Inamura in compiling his dictionary. He afterward revised and enlarged his father's work on medicine, and also published in 1806 a book called *I Han Teikô,* or "Manual of Medicine," which was of great service in diffusing Western medical knowledge. His mastery of Chinese made him a ready writer and translator—although, indeed, this might be said of almost all of those early pioneers of the new school.

Yoshida Chôshuku (1779–1824), a pupil of Katsuragawa, being led to the study of original Dutch books by reading Udagawa's *Naika Sen-yô,* was the first to begin the open practice of Dutch medicine. This gave great offense to the doctors of the old or Chinese school, who insisted that the Dutch system should be confined to surgery, as heretofore, and denounced the new medicine as outlandish and vicious; so that Katsuragawa was obliged to scratch Yoshida's name off the list of his pupils. Yoshida, however, was very successful, and afterward, on the recommendation of Udagawa, became a physician to the Lord of Kaga. He

published in 1814 a book on the treatment of fever, entitled *Taisei Netsubyô Ron,* with a later supplement, and also a work on Dutch *materia medica.* He had many pupils—among others, Takano Chôei and Koseki San-ei.

Yoshiwo Jôan was the first to call attention to the importance of the study of physics, and as an introduction wrote a book on celestial phenomena called *Kwanshô Zusetsu* (1823). Aochi Rinsô (1775–1853) was the first to publish a book on physics, *Kikai Kwanran* (1827), which was afterward amplified by Kawamoto Kômin (1810–1871) in his *Kikai Kwanran Kwôgi* (1851). Kawamoto was interested in applied science, and made various experiments; he was successful in taking daguerreotypes and photographs. Aochi's *Bankoku Yochi Shiryaku* may also be regarded as the first systematic book on geography, although unfortunately it was not printed. In 1833 was published *Shokugaku Keigen* by Udagawa Yôan (1798–1846), adopted son of Udagawa Genshin, containing an exposition of systematic botany after Linnæus; and in 1839, *Seimi Kaisô,* by the same author, which was the first book on chemistry.

We have already seen that the shôgun Yoshimune was interested in astronomy and founded an observatory. Astronomy, however, did not flourish; the knowledge of Western astronomy and mathematics, transmitted by Hayashi through Kobayashi to Nishigawa, died out with the last-named scholar. There were attempts at the translation of books on astronomy, such as that by Motoki Nidayû, a Nagasaki interpreter, who was ordered to translate a book on the use of globes, and notwithstanding his ignorance of the subject did accomplish the task (1793) after toiling at it for two years. The truth is that while in medical and allied sciences the translators were doctors who had some knowledge of the subject, or at all events were animated by

a zeal for it, astronomy suffered from an utter lack of mathematical knowledge on the part of those who understood Dutch. It may here be stated that a system of mathematics was being developed in Japan quite independently of Western mathematics, which was not introduced till later on, and even then it was cultivated side by side with, but quite distinct from, the latter. Under these circumstances the Observatory had fallen upon evil days, and the almanac for 1795 failed to predict the total solar eclipse which took place on New Year's day (old calendar). A reform was now imperative, and Asada Gôryû (1734–1799) was summoned from Osaka to take charge of the task. He was a man of great originality; a physician by profession, he had devoted himself to astronomy and had made observations with instruments made by himself, and arrived independently at several important results, which he afterward found to agree with those of Western astronomers as stated in Chinese books (translations or compilations mostly by Catholic missionaries in China). Asada was too old to come to Yedo himself, but sent his two pupils, Takahashi Sakuzaemon (1764–1804) and Hazama Gorobei (1756–1816), in his place. They were both men of great ability, and under their direction a revised almanac was issued for 1798. Hazama then went back to Osaka. He was a man of some means, always had artisans working for him, and among other instruments made a barometer and a thermometer, with which he began meteorological observations which were kept up for some time after his death; he also devised an ellipsograph which is described by his son. The instruments used by Inô in his survey were made under the direction of Hazama after European models. Takahashi, Asada's other pupil, was placed permanently on the staff of the Observatory. It was at his suggestion and under his superintend-

ence that the geodetic survey of Japan was undertaken by Inô Kageyu (1744–1818). Inô was well over fifty when he began the survey in 1800, and spent the rest of his life on the survey, so that the maps were almost complete at the time of his death. The wonderful accuracy of these maps, which are still preserved and parts of which have continued to be the standard map down to the present day, bears ample testimony to the skill, patience, endurance, and scientific conscientiousness of Inô. Takahashi did not live to see the completion of Inô's survey; he died in 1804, and was succeeded by his son Takahashi Sakuzaemon, junior (1785–1830), also an able and enterprising man.

At the suggestion of Takahashi, junior, a bureau of translation was established in 1811 in the Observatory, Otsuki Gentaku, Baba Sajûrô (1787–1839), a Dutch interpreter of Nagasaki, and Udagawa Genshin being the earliest members of the staff, which included at one time or another most of the eminent Dutch scholars, such as Otsuki Genkan (1785–1837), son of Gentaku; Udagawa Yôan; Sugita Rikkei (1786–1845), son of Genpaku; Sugita Seikei (1817–1859), son of Rikkei; Aochi Rinsô; Koseki San-ei (1787–1839); Mitsukuri Genpo (1799–1863), grandfather of the present writer; Kawamoto Kômin; etc. This bureau of translation was the germ which has developed through several stages of transformation into the present Imperial University of Tokyo. Such a bureau was decidedly a desideratum at that time; for the Russians in the north and the English in the south were beginning to make their presence felt, sometimes in a very unpleasant manner, and the government was desirous of obtaining a fuller and more accurate knowledge of the outside world. Already Dutch scholars had written many books and pamphlets, giving information concerning the nations of the world, of which some were printed and

published, some circulated privately in manuscripts, and some kept secret for official or individual reasons. Such were the *Bankoku Zusetsu* ("Map of the World, with Explanations," 1786) by Katsuragawa, the *Bankoku Shinwa* ("New Talk about Different Countries," 1789) by Morishima (a brother of Katsuragawa), a revision and enlargement of Arai's *Sairan Igen* by Yamamura Saisuke (a pupil of Otsuki, 1802), *Ho Ei Mondô* (a warning about the movements of the English, 1807 and 1808) by Otsuki Gentaku, etc., besides many books on Russia and the Russians by almost every one of the above writers and several others. (I mention these to show that those pioneers of the new learning were alive to the dangers of foreign attack, and were the first to warn their countrymen of it.) In 1808 several of the interpreters at Nagasaki were ordered to learn Russian and English. One of them, Motoki Shôzaemon, wrote an English grammar (1811) and compiled an English-Japanese dictionary (1814), neither of which was, however, printed. It was not till 1847 that the study of English began to be taken up seriously in Yedo. About this time Rin Shihei (1738–1793) traveled all over Japan from Yezo to Nagasaki, and became convinced of the pressing necessity of coast defense, and of the danger arising from its total neglect. He tried to impress upon his countrymen the magnitude and imminence of this danger, and with this object he wrote several books, among others *Kaikoku Heidan,* or "Talk on the Arms of an Island Country" (1787, published 1791). This book led to his being kept in confinement (1792) for trying "to excite the people to unnecessary unrest by publishing preposterous opinions based on ridiculous rumors."

The arrival of Philipp Franz von Siebold as physician to the Dutch factory was a great event in the history of the

introduction of Western knowledge; for, besides his exceptional skill in medicine, he was also well equipped scientifically for carrying on the investigations in natural history for which he had come to Japan. He resided for six years, from 1823 to 1829, in Nagasaki, where he gave clinical lectures, and many Japanese doctors and scientists visited him and greatly profited by his instruction and guidance, while he himself also derived immense advantages from their assistance. In 1826 he came to Yedo, where, among others, Takahashi of the Observatory became acquainted with him, and gave him a map of Japan in exchange for some books which Takahashi was most anxious to acquire as likely to give a very good idea of the state of Europe, but which Siebold would not give him on any other condition. Now it was against the law to give a map of Japan to a foreigner, and this act of Takahashi being afterwards discovered, he was thrown into prison, where he died soon after. At the same time an ophthalmologist, Habu Genseki, was severely punished for having given Siebold, in exchange for some ophthalmological books and instruments, a kimono with the shôgun's crest which had been given him as a reward for some special service. Many others suffered in connection with this, and Siebold himself was expelled from the country. This was a very unfortunate occurrence, for Siebold had been a great help to the students of Western learning, and his expulsion was a real blow to its cause, and this act of disloyalty, even though it had been done with good intention, brought reproach on the votaries of the new learning.

Among those who received Siebold's instruction in Nagasaki were Itô Keisuke, Itô Genboku, Totsuka Seikai (a pupil of Udagawa Genshin), Takano Chôei (a pupil of Yoshida), and others. Itô Keisuke (1803–1901) became an eminent botanist, and in 1901 was raised to the peerage at the age of

ninety-eight for his services to the state as scientist. Itô Genboku (1800–1871) and Totsuka (1799–1876) came to Yedo and practised, taught, and wrote books on the Dutch medicine. They were very successful, Itô afterwards being appointed physician to the Lord of Hizen (1844) and later to the shôgun (1858), and Totsuka to the Lord of Satsuma (1842). Takano Chôei (1804–1850) was a man of great talent, a very good Dutch scholar, and a facile writer and translator; he also came to Yedo (1830) and began to practise and teach medicine; he translated many books, among which his *Igen Sûyô,* a work on physiology, deserves to be specially mentioned here. But his active nature and independent spirit did not allow him to lead a quiet life. With his friends, among whom the most prominent were Watanabe Noboru (1794–1842), chief adviser to a small daimyô; a Chinese scholar and artist (well known by the *nom de plume* of "Kwazan"), who, although not himself a Dutch scholar, was convinced of the importance of Western learning; and Koseki Sanei (1787–1839), already mentioned as a pupil of Yoshida,—with these and others, Takano held periodical meetings, at which they discussed all sorts of topics, literary, scientific, social, industrial, and political, in the light of Western knowledge. One day, hearing that the government had decided to send away, by force if necessary, an English ship if it should appear in Yedo Bay in accordance with the information given by the Dutch, they earnestly discussed the subject and came to the conclusion that those who understood the condition of the outside world should not be silent on such an important occasion. Accordingly, Takano wrote a brochure called *Yume Monogatari* ("A Dream"), in which he urged, in the words of a man met in a dream, the unadvisability of such a policy. This brochure

was presented to the officials of the shôgun and secretly circulated among Takano's friends. Watanabe also wrote some notes which he, however, with his natural modesty and prudence, kept to himself.

It was to be expected that the conservative element, among whom we may count the Chinese scholars in general, would look with no favorable eyes upon the instruction in what they regarded as barbarian and outlandish. One of the most persistent and implacable of them was Torii Yôzô, a narrow-minded man, who had special reasons to be unfriendly to the advocates of the new learning. He was a cadet of the Hayashi family, whose head was hereditary doyen of the Chinese literati, and on one occasion, as the head of a commission to make a survey of the coast of Izu and Sagami and to report on the best means for its defense, he had the mortification of seeing the report of Egawa Tarozaemon, his second on the commission, accepted in preference to his own. This Egawa was a friend of the new learning, and had the assistance of Uchida Yatarô and Tamura Kisaburô, pupils of Takano Chôei, who were acquainted with the modern method of surveying. The patriotic but somewhat too ardent and imprudent zeal of Takano and others gave Torii a good opportunity of taking a personal revenge and at the same time of arresting the advance of the new movement. Watanabe was ordered to be kept in confinement in the domain of his lord, where he afterwards committed *hara-kiri,* having reasons to fear that his living might be prejudicial to the interests of his lord. Takano was put into a common prison, whence he escaped at the time of a fire, and after being in hiding for some time, during which he was employed in making translations, was discovered and killed himself in order to avoid further

humiliation. Koseki Sanei also killed himself as soon as he heard of the arrest of Watanabe and Takano, and many others suffered in various degrees.

Another victim of Torii's enmity was Takashima Shirodayû of Nagasaki, who, having learned modern gunnery from a Dutchman, had been summoned in 1841 to Yedo to exhibit his method and skill. Egawa Tarozaemon was the first to enroll himself as his pupil and to receive instruction in the new method. After his return to Nagasaki, Takashima was accused of secret intercourse with the Dutch and thrown into prison, whence, however, he was released in 1853 to give instruction in gunnery.

The way of Dutch scholars, which had been by no means smooth before these events, was now made still rougher by various restrictions, which, however, could not stop the steady progress of Western knowledge. Among the pupils of Udagawa Genshin were Tsuboi Shindô (1795–1848), Mitsukuri Genpo (already mentioned), and Totsuka Seikai. Tsuboi began to systematize the teaching of the Dutch by prescribing a course in which the reading of grammar had an early and important place. One of his pupils, Ogata Kôan (1810–1863), began to practise the Dutch medicine and to teach the Dutch language and medicine in Osaka in 1838. Ogata's school, which was in existence till 1862, and of which a most interesting and vivid account is given in the autobiography of his pupil, Fukuzawa Yukichi, the founder of the *Keiô Gijuku,* became the center of Western learning in western Japan, and counted over three thousand pupils, among whom were many leaders of new Japan, too numerous to mention. Another pupil of Tsuboi, Sugita Seikei, in Yedo also had many distinguished pupils, among whom may be mentioned Kanda Kôhei, who first taught Western mathematics in the *Kaiseijo,*[1] and Sugi Kôji, the father of

[1] See page 704.

statistics in Japan. Books on law and politics were now ordered to be translated in the Translation Bureau, though solely for official use. Mitsukuri Genpo wrote the *Taisei Shinjû,* the first systematic history of Europe; while his pupil and adopted son, Mitsukuri Seigo, published his *Konyo Zushiki* (1847), which gave the general public for the first time a tolerably up-to-date knowledge of the geography of the world. Mitsukuri also printed a Dutch grammar in script characters by means of wood blocks (the usual way in those days), which was a great help to the students of Dutch, for before this they had to copy the book for themselves before beginning to read it. This continued to be the case with most foreign books until well on in the sixties, for imported books were scarce and they could not be printed in Japan; the present writer did not have to do this copying, but he can remember his brother, elder by a few years, copying (somewhere about 1866) Markham's "History of England," which he was learning to read. Fujii Saburô was the first to attempt the reading of English books, and his *Ei Bun Pan* was the first book published on the subject (1847). About this time, also, Murakami Eishun (1811–1883) for the first time began to read French books with the help of a French-Dutch dictionary.

By the middle of the nineteenth century doctors practising Dutch medicine had become so many and so successful, especially in Yedo, as to cause serious uneasiness to doctors of the old Chinese school; and through the influence of the latter an injunction was issued in 1849, confining the practice of the Dutch school to surgery only, so that Itô Genboku and others had to enroll themselves pupils of Katsuragawa, the shôgun's surgeon, before they could practise publicly. Moreover, it was made necessary to obtain the permission of the authorities of the old Medical Academy before pub-

lishing any book on the new medicine: this of course was tantamount to a prohibition. It was not much better with books other than medical: permission to publish any work relating to Western learning was always granted very grudgingly; thus, for instance, my grandfather, although he was on the staff of the Translation Bureau, had to wait for two years (from 1849 to 1851) after the wood blocks had been completed before he could get permission to publish his *Hakkô Tsûshi,* a book on geography. But even such measures were not sufficient to stop the introduction of Western learning, and the coming of the American, Russian, and English ships demanding the opening of Japan to trade, and the subsequent change of policy on the part of the shôgun's government, made the knowledge of foreign languages and foreign matters in general imperative.

In looking back over this period, the first thing that strikes us is the fact that the first introduction of Western knowledge was almost entirely due to doctors of medicine, who, however, as we have seen above, did not confine themselves to medicine alone. This was due to various circumstances. As I have remarked before, about the middle of the eighteenth century there arose in Japan a remarkable revolutionary movement in things intellectual, a general restlessness and reaction against old authorities, a search for new knowledge; and the doctors were almost the only persons possessing sufficient culture who were likely to turn their attention to foreign learning. Moreover, the superiority of the Dutch method in surgery had long been acknowledged, and their superiority in other branches of medicine could also be demonstrated by facts and appreciated by the public; and thus this was the door through which Western learning could enter with the least resistance.

I have perhaps not stated explicitly enough the difficulties

and dangers confronting those who were bold enough to break through the hard crust of custom and prejudice and to attempt to learn a strange language and so to open an avenue to a new and alien learning; to do so would require too long a digression into the organization of the society and the character of the civil administration of the time; suffice it to say that they were very great, indeed, and sometimes insuperable.[1]

Special mention should, however, be made of the assistance that many of the daimyôs, actuated some by true and intelligent perception of the importance of the new movement, others by mere curiosity or vanity, rendered to its pioneers by their patronage and by giving them leisure to pursue their study, as well as by supplying them with books and other materials.

THE FOURTH PERIOD

INTERESTING as it would be, this is not the place to describe the stirring events which followed the coming of Commodore Perry in 1853 and the opening of the country again to foreign intercourse, and led to the "Restoration of Meiji" in

[1] I cannot refrain from mentioning one example of these difficulties. Even toward the end of this period, when it had become comparatively easy to get Dutch books, it was only through the shôgun's officials, and with their permission, that a private individual could obtain a foreign book, and then not more than one a year. Often interpreters who accompanied the Dutch chief factor from Nagasaki on his visit to Yedo brought some books with them which they sold secretly to the Dutch scholars at a great profit. In one of my grandfather's (Mitsukuri Genpo) letters to my father (Mitsukuri Shûhei), he complains that the Dutch, having met with a theft on the way, were so strictly guarded that it was impossible to get an interview with them as usual, and that the interpreters were afraid of selling the books that they had brought, or else demanded such prices for them that a poor scholar like himself could not afford to buy. Yet this very difficulty was often an incentive to a new line of study; as an instance, I may mention the case of Murakami, who, failing to get the Dutch book on chemistry that he wanted, but being supplied with a French book in its place, set to work to learn to read French instead of waiting for the Dutch book, which would be at least eighteen months in coming.

1868; we must confine ourselves to those relating more particularly to the subject in hand.

In 1855 the Translation Bureau was made independent of the Observatory, and under the name of *Bansho Shirabejo* ("An Institution for the Study of Foreign Books"), which was finally changed to *Kaiseijo,* besides translation, instruction was given in foreign languages, not only to the shôgun's immediate retainers but also to those of daimyôs, Mitsukuri Genpo and Sugita Seikei being among the earliest professors. The foreign languages taught were Dutch, English, Russian, French, and German. A department of natural products (or natural history) was added in 1861, with Itô Keisuke as professor; a department of mathematics (although naturally of an elementary character) in 1863, with Kanda Kôhei as professor; and a department of physics and chemistry in 1865, under a Dutch professor named Gratama. In 1867 the modern method of class teaching was introduced.

In 1863 a foreign language school was opened in Nagasaki by the shogunate, at which Chinese, Dutch, English, French, and Russian were taught. Thus the instruction in foreign languages hitherto given only by private persons was now given at those schools or academies by professors appointed by the government of the shôgun. Some of the greater daimyôs followed the example and established schools for the teaching of one or more foreign languages, usually English, which now came to be studied more than any other language—more even than Dutch. At the same time private tuition went on as before, and some regular private schools were established, of which that of Ogata, already mentioned, and that of Fukuzawa, afterward called "the *Keio Gijuku,*" were the most notable examples.

The march of events was such that the injunction against

the practice of Dutch medicine lost its effect. In 1857 Itô Genboku, Totsuka Seikai, and others opened a "vaccination institute," where doctors of the new school held meetings, there being more than eighty of them in Yedo at the time. Next year Itô and Totsuka were called in to attend upon the shôgun in his illness. The Vaccination Institute was made a government institution, with three departments for instruction, for discussion, and for vaccination. In 1861 the name was changed to *Seiyô Igakujo* ("The Academy of Western Medicine"). In 1860 Matsumoto Ryôjun opened a hospital in Nagasaki, where he had been studying under a Dutch naval medical officer named Pompe. The next year this hospital was turned into a government school of medicine, with a Dutch doctor named Bowdoin as professor; this doctor was the first foreign professor employed by the Japanese government. In 1865 physics and chemistry were added to the subjects taught in this institution.

Missionaries now began to come to the open ports and gave lessons in languages; some were engaged by daimyôs to teach in the interior. Among the missionaries the names of the Americans Hepburn, Brown, and Verbeck must specially be mentioned, all men of sterling character and attainments. Dr. Hepburn practised medicine in Yokohama; his Japanese-English dictionary, the first of its kind, is still in use, and the system of transliteration of Japanese characters into the Latin alphabet employed in it has remained the standard down to the present day.

Books, translations, and original works on various topics now become too numerous to enumerate; I shall mention only two besides Hepburn's dictionary: one is the English-Japanese dictionary compiled by Hori Tatsunosuke, assisted by teachers in the *Kaiseijo,* and the other the work entitled *Seiyô Jijô,* or "Things Western," of Fukuzawa Yukichi, in

which he describes what he had observed of the Western world during his travels in America and Europe, whither he went as a translator to the embassies sent by the shogunate to America in 1860 and to Europe in 1861. This book did more to make the West known to the general public than almost any other book; indeed, it was unique at the time both in the nature of its contents and in the number of copies sold.

In 1862 the shôgun's government sent a number of students to Holland, among whom were Enomoto (afterward Viscount, Minister of the Navy, of Education, etc.) and Akamatsu (Admiral, Baron), to learn navigation; Itô Genpaku and Hayashi Kenkai to study medicine; Nishi Amane and Tsuda Mamichi, who studied law (both afterward barons). The next year four students were sent to Russia. In 1866 a party of fourteen students was sent to England, among whom were Nakamura Masanao, already known as a Chinese scholar, and afterward a great educationalist; Toyama Masakazu (afterward professor and president of Tokyo University, and Minister of Education); Hayashi Tadasu (Count, the present Minister of Communications); and the present writer, the youngest of the party (being eleven years old at the time), with his elder brother, Mitsukuri Keigo. Finally, in 1867, the shôgun's brother, Tokugawa Minbutayû, was sent to France with another party of students: in his suite were such men as Shibusawa Eiichi (now Baron) and Mitsukuri Rinshô (afterward Baron, grandson of Genpo). A few of these students came home before the Restoration, but all were recalled in 1868. Most of them afterward did good service in the introduction of Western learning into Japan. The Satsuma clan also sent a number of students abroad, and a few went on their own initiative, among whom were the late Prince Itô and Mar-

quis Inouye: these had to go secretly, as the order forbidding all traveling abroad was still in force.

Although the shôgun's government saw the necessity of opening the country to foreign intercourse, the conservatives all over the country were bitterly opposed to such a step. This opposition to the foreign policy of the shogunate, inseparably combined with the more fundamental one based on our national constitution, namely, that the shôguns were usurpers and were wielding authority which properly belonged to the Emperor alone, was the force that ultimately brought about the downfall of the shogunate and the "Restoration of Meiji." Conservative feeling ran very high, and masters of the new learning were now often in danger of their lives from conservative samurais, who regarded their action as a desecration of the land of the *Kami* (ancient gods of Japan). Sakuma Shuri was assassinated in Kyoto for his open advocacy of the opening of the country. It was under the cry of "Reverence for the Sovereign!" and "Exclusion of Barbarians!" that the overthrow of the shogunate was effected.

THE FIFTH PERIOD

We now come to the era of *Meiji,* or "The Enlightened Government," which began in 1868 and ended with the death of Emperor Meiji in July of the present year (1912). The accession of the Emperor took place in the beginning, and the resignation of Keiki, the last of the shôguns, toward the end, of the preceding year. A few disaffected followers of the shôgun took up arms against the imperial banner, but were put down without very great difficulty, and thenceforth the Emperor reigned in fact as well as in name. Although the cry for the overthrow of the shogunate had been "Reverence for the Sovereign!" and "Exclusion of Barba-

rians!", yet the leaders of the movement knew well that the last was neither practicable nor desirable; and on the fourteenth day of the third month of the first year of Meiji (April 6, 1868), the Emperor summoned the imperial princes and high officials of his court, and in the *Shishinden,* or throne-room, of the old palace in Kyoto swore the memorable oath known as "The Imperial Oath of Five Articles," setting forth the policy which was to be followed by him thereafter. The five articles were as follows:[1]

I. Deliberative assemblies shall be established, and all measures of government shall be decided by public opinion.

II. All classes, high and low, shall unite in vigorously carrying out the plan of government.

III. Officials, civil and military, and all common people shall, as far as possible, be allowed to fulfil their just desires, so that there may not be any discontent among them.

IV. *Uncivilized customs of former times shall be broken through,* and everything shall be based upon just and equitable principles of nature.

V. *Knowledge shall be sought for throughout the world,* so that the welfare of the Empire may be promoted.

In pursuance of the policy set forth in the above oath, the first ten years of the Meiji era were occupied mainly in breaking up the established order of things and substituting a new one; although, as for the latter, a much longer period elapsed before anything satisfactory could be arranged. Many great and radical changes were made, of which the

[1] The translation is that of Dr. Hozumi Nobushige, Emeritus Professor of Law in the Imperial University of Tokyo.

greatest by far was the abolition of the feudal system, which was completed in 1871: the daimyôs, or great military lords, gave up, of their own free will, all their lands and the power of life and death over their retainers and people within their respective territories, receiving in compensation pensions which were afterward commuted into national bonds. A new system of civil administration was introduced, and laws were revised. The wearing of swords by samurais was forbidden, the army and navy were reorganized, and a system of universal conscription elaborated, so that the samurais, or military class, no longer were allowed to monopolize the civil and military services.

Schools established by the shogunate and closed at its overthrow were reopened as soon as order was restored, and many new schools were opened both by the central and the local government (those of the daimyôs before the abolition of feudal clans). Many private schools for the teaching of Western knowledge flourished, among which may be specially mentioned the *Keiô Gijuku* of Fukuzawa, the *Sansa Gakusha* of Mitsukuri Shûhei (father of the writer), and the *Dôninsha* of Nakamura Masanao. Of Fukuzawa it is related that in May, 1868, while fighting was going on in Ueno (now Ueno Park, Tokyo) between the imperial army and some retainers of the shôgun, Fukuzawa continued to hold his classes in another part of the city, and his school was not closed for a single day.

In 1872 the first Education Code was promulgated, by which a national educational system was introduced for the first time. According to this, the whole country was to be divided into 8 university districts, each with a university; each university district was to be subdivided into 32 middle school districts, each with a middle school; and each middle school district was again to be subdivided into 210 elemen-

tary school districts, each with an elementary school, so that there would be 8 universities, 256 middle schools, and 53,760 elementary schools in the whole country: the elementary school education was to be compulsory for all classes and both sexes. At the same time as the promulgation of the new code, all existing schools supported by the government, central or local, were to be reorganized so as to be brought into conformity with its provisions or else be closed. The scheme of the code, however, proved too ambitious to be carried out in its entirety. In fact, in this, as in many other forms that followed the Restoration, we began with copying too closely the system or model of some one country, and that not always the one best suited to our circumstances, sometimes trying one model after another in our effort to find out what was the best; but gradually, as our knowledge has increased and our field of vision become widened, we have tried to adapt and make it more suitable to our own needs, by a careful consideration not only of systems and methods of different countries in theory and practice, but also of our own customs, usages, and traditions, and the peculiar circumstances of the times, which at first were often overlooked.

We cannot go afield into the whole question of the educational system, but must confine ourselves to the introduction of Western learning. Before the coming of Commodore Perry this was naturally most easily effected through the medium of the Dutch language, which, indeed, may be said to have been the only channel then available. But with the opening of the country to foreign intercourse, the English language began to be more generally studied, as it was the current language of the East. American missionaries helped to spread the knowledge of it among the Japanese people, many of them becoming teachers in schools after the Resto-

ration. The study of foreign languages in general, which had presented such great difficulties and even dangers in the earlier days, was now stripped of all extraneous difficulties and encouraged and made a part of the higher common education, so that from that time on mere study of foreign languages scarcely comes within the scope of our subject. In private schools for foreign languages, however, students were often of mature age and had had previous culture in Chinese literature; they read works on politics and economics, on Western philosophy and other abstruse subjects, as well as books on history, geography, and other common subjects, with a view to mastering the subject-matter, and consequently a knowledge of those subjects became more general. Gradually, as higher common education spread, and with it the study of English, these private schools lost in large part their *raison d'être,* and in the eighties most of them were either closed or transformed partly or wholly into middle schools for higher common education, or into colleges for the teaching of special subjects.

In the *Kaiseijo* (Academy for Foreign Languages) established by the shogunate and reopened by the new government, the same kind of tuition as in private schools was carried on by Japanese teachers for some time, side by side with the new and systematic instruction in foreign languages under Japanese and foreign teachers; but soon the former part was discontinued, and, on the other hand, provisions were made for instruction in law, some branches of science and engineering, and in history, philosophy, and literature, with a view to make it a nucleus for a university. In 1877 the *Kaiseijo* and the *Igakujo* (see pages 705, 715, and 717) were incorporated as the University of Tokyo, with four faculties of law, science, literature, and medicine, to be again reorganized in 1886 into the present Imperial University of

Tokyo (by amalgamation with the Engineering College, formerly under the Department of Public Works), with five "colleges," or faculties, of law, medicine, engineering, literature, and science, to which was afterward (1890) added a College of Agriculture. Let us now briefly consider the development of these faculties or colleges.

Before the Meiji era scarcely any attention had been paid to Western laws and political science; the few books on these subjects that had been translated by order of the shôgun's officials had not been made public, it being the policy of the shogunate to suppress all political discussions as much as possible. With the Restoration all this was changed. The reorganization of civil administration and the revision of laws and legal procedure required a knowledge of Western facts and ideas on those subjects, and books bearing on them began to be eagerly studied in the original or in translations. Accordingly, those who had acquired some legal knowledge of the West, such as Tsuda Mamichi, Nishi Amane, Mitsukuri Rinshô, and others, were in great demand. A translation of the Code Napoléon made by the last named was an important work, and contributed greatly to the spreading of the knowledge of Western legal ideas. In 1873 a French legal expert, M. Boissonade, was engaged as adviser to the Department of Justice.

It is not the province of this paper to trace the history of the codification of Japanese laws, which occupied a period of some forty years, but it may be briefly stated that the first draft, a close copy of the French code, was considerably modified through a greater attention paid to the old and established customs and usages of the country, and by the taking into consideration of the laws of other lands, especially of Germany. In this we have another very good instance of what we have stated above in connection with the

educational system. The names of Professors Hozumi Nobushige, Tomii Masaakira, and Ume Kenjirô, of the Imperial University, Tokyo, must be mentioned even in this brief notice; for to them and to Mitsukuri Rinshô more than to any others is due the credit of the successful accomplishment of the work of codification.

A school was opened in 1872 under the Department of Justice to give instruction in French law, while in the *Kaiseijo* a course in English law was opened in 1874, as stated above. We find in the calendar of Tokyo University for 1878 three professors of English law, one Englishman, one American, and one Japanese, the American being Professor H. T. Terry (Yale, '69), who has just retired this summer (1912), and the Japanese, Inouye Ryôichi, one of the first two Japanese graduates of Harvard Law School. There were also some lecturers on old Japanese laws. In 1885 the school of French law was transferred to the university, and in 1887 a course of German law was added. As the work of legislation progressed, lectures on Japanese law were given at first as auxiliary subjects, but finally they came to be the main subjects, while lectures continue to this day to be given on English, French, and German law as auxiliary subjects. Public laws, political sciences, and economics also now form a part of the curriculum of the Law College, which at present consists of the four sections of law, politics, economics, and commerce. I cannot do better than sum up by quoting Professor Tomii's remarks: "Thus the two decades immediately subsequent to the Restoration were characterized by prevalence of the study of French, English, and American laws. . . . But times changed. The past twenty years have witnessed the rise and ascendancy of German law, and a tendency has grown up to take it as the model in studying jurisprudence and legislative work, whether in the domain

of public or of private law. . . . Recent developments have been remarkable, and the stage of imitation has already been left behind." ("Fifty Years of New Japan," by Count Okuma.) These remarks will apply also to political and economic sciences, as indeed to almost all branches of learning introduced from the West.

Early in the eighties, owing to changes in civil administration and in laws and legal procedure, there was felt a great want of men having special knowledge of these subjects, and the single University of Tokyo not being able to turn out a sufficient number of such men, several colleges were started by private individuals, who disinterestedly gave some of their leisure hours to teaching in them; the first of these was the *Senshû Gakkô,* opened in 1880 by Tajiri Inajirô (a Yale graduate) and others to give instruction in law and economics. This was followed within a few years by many others, among which was the *Waseda Senmon Gakkô* of Count Okuma. The *Keiô Gijuku* also changed its organization so as to have college courses in law, political economy, and literature. In Tokyo University itself a special course was organized temporarily, in which instruction was given in Japanese for those who had not passed through the preparatory course, so as to enable them to follow the regular course of lectures. It may be mentioned here that in almost every subject lectures in the university were given at first in some foreign language (German in the case of medicine, English in others), not by foreign professors alone, but by Japanese professors as well; for it was very difficult to find proper translations not only of technical terms, but also for necessary technical expressions and phrases, these being even more troublesome than simple terms on account of the peculiar nature of the Japanese language. Indeed, one of the initial difficulties in the intro-

duction of Western learning may be said to have lain in the difficulty of translation, our language being so radically different in its structure from European languages. Thus the lectures in Japanese to special classes served the double purpose of turning out a large number of moderately well trained men, and of giving professors a good exercise in lecturing in Japanese on technical subjects. The opening of such special classes in the university for a time was not confined to the law faculty, but was found necessary in other faculties also. However, to return to private colleges, the maintenance of such is somewhat difficult in Japan, as no large fees can be charged owing to the poverty of most of the students, and endowments such as are so common in America cannot be expected, those even of *Waseda* and *Keiô* being quite insignificant in comparison with the endowments of even smaller colleges in America. In those earlier days of the Meiji era, when the number of students was small, most of the founders were themselves teachers who gave their time and services free, besides in many cases contributing to the expenses of maintenance. For this reason, there are but very few private colleges of medicine, science, or engineering, their establishment and maintenance being too costly to be supported by fees. I may mention incidentally that most of these private colleges have now assumed the more ambitious title of universities.

As the introduction of Western learning previous to the Meiji era had been due almost exclusively to doctors of medicine, although happily they did not confine their attention to medicine alone, it was natural that at the outset more progress should have been made in medicine than in other subjects, and it was in medicine that systematic instruction was first introduced after the Restoration. The *Igakujo* was one of the schools reopened by the new government,

and with it was incorporated a hospital newly opened by the government under the direction of an English surgeon, Dr. Willis. The government, however, having decided to Germanize medical education, Dr. Willis left the hospital and went to Kagoshima, where until 1877 he taught in a medical school with great success. Meanwhile two German doctors, Müller and Hoffmann, were engaged in the *Igakujo* in 1871, and organized a system of medical instruction consisting of a five-year preliminary or general course and a five-year special or medical course. Almost all the professors and teachers, including teachers in German, Latin, and elementary mathematics, had to be brought from Germany. As the number of those who could enter this regular course of ten years was limited, owing to the lack of accommodation and equipment, while on the other hand the demand for doctors of the Western school was great and insistent, a short special medical course was opened, in which instruction was given in Japanese by Japanese professors. In the calendar for 1877 we find the names of eleven German professors and teachers, besides seven Japanese professors engaged in teaching the students of the short course. This course was afterward discontinued, as several colleges of medicine came to be established in different parts of the country to carry on a similar work. The College of Medicine in the university itself has gradually grown to be a large body with twenty-seven professors, all Japanese, including four in pharmacy, and nineteen assistant professors and lecturers, and nearly eight hundred students.

With regard to science and its application, we have seen that translations of books on various scientific subjects had been made by Dutch scholars, some of the more important of which we have mentioned above. But there must have been many that were not printed or even privately circulated,

for there are in possession of the writer's family translations of works on astronomy, geology, mineralogy, etc., left in manuscript by Mitsukuri Genpo, and no doubt there were similar manuscripts left by others. In Western mathematics, physics, and chemistry, teaching of the elementary parts was begun in the *Kaiseijo* before the Restoration, as already stated, but it was not revived for some time after the school was reopened. In astronomy such practical knowledge had been introduced as was necessary for the compilation of almanacs. In natural history some advance had been made in systematic botany. As for applications of science to practical purposes, but little knowledge had been introduced.

On the promulgation of the first Education Code, the *Kaiseijo* was made a middle school, the instruction being given in a foreign language (English, French, or German), mostly by foreign instructors. Soon after courses were opened in special subjects, of which the one in English law has been already noticed. The other courses were those of physics, chemistry (pure and applied), mining and metallurgy, civil and mechanical engineering, and literature and philosophy. In the calendar for 1876 we find eighteen foreign professors and instructors, including two professors of English law. The incorporation of the *Kaiseijo* and the *Igakujo* into the University of Tokyo in 1877 gave a great impetus to the study of science. Mathematics was made one of the main subjects (previously it had been merely an auxiliary subject for engineering students), and the study of its higher branches was entered upon. The appointment of Dr. Fujisawa Rikitaro in 1888 as professor of mathematics in conjunction with the present writer gave a new impetus to the study of higher mathematics. The year 1877 saw the foundation of the Tokyo Mathematical Society, which is the first of many scientific societies now existing, and which has

since developed into the present Tokyo Mathematico-physical Society, holding monthly meetings for the reading of original papers on mathematics, astronomy, and physics, and publishing them (in Japanese, English, or German) in its proceedings and transactions.

In physics the coming of Professor Mendenhall (afterward superintendent of the United States Coast and Geodetic Survey) marks the beginning of the teaching of experimental physics and of original investigations. He was succeeded by Professor Ewing, whose work on hysteresis was begun in Japan; and their work has been ably carried on by their pupils and successors, Tanakadate Aikitu, Nagaoka Hantaro, and others. Instruction in practical astronomy was started by Professor Paul, of the United States Naval Observatory, who was succeeded by Professor Terao; and although from its nature astronomy does not possess many votaries in Japan, and although the university observatory is at present but poorly equipped, Japanese astronomers have made some contributions to the science, as, for example, in the observations of variations of latitude, for which an international observatory has been established in Mizusawa and placed under the direction of Dr. Kimura, whose discovery of the z-term in the equation of the variation of latitude has recently been awarded a prize by the Imperial Academy of Tokyo. In chemistry, pure and applied, we had Professors Atkinson (English), Wagener (German), and Jowett (now of Oberlin College), whose places were not long after taken by the Japanese professors, Sakurai Jôji and Matsui Naokichi: the former still occupies the chair of chemistry in the Imperial University, and during his long career of over twenty-five years in the university has contributed both by his teaching and original researches not simply to the introduction of that science into Japan, but to

the advance of the science itself; while the latter, too, did great service not only in the introduction of chemistry, but also of scientific agriculture in his capacity as director of the College of Agriculture from its amalgamation with the university in 1890 to his death in 1910.

In natural sciences, Dr. E. S. Morse, of Salem, Massachusetts, came in 1877 as professor of zoölogy; he established the first zoölogical laboratory in the university, and was also the first to expound to the Japanese public, by a series of public lectures, the Darwinian theory of the origin of species and the descent of man. He was succeeded by Professor Whitman, late of Chicago University, after whom the chair was occupied by Dr. Mitsukuri Kakichi (brother of the writer), supported by his colleague, Professor Iijima Isao, who had been a pupil of Whitman, and afterward of Leuckart in Leipsic. The chair of botany was occupied from the first (1877) by a Japanese, Yatabe Ryôkichi, a graduate of Cornell, with Dr. Itô Keisuke, then over seventy years of age, as honorary professor. To these men is due the credit of having introduced into Japan modern methods in biology, the elements of which now form a part of the curriculum of common education.

Geology, mining, and metallurgy also began to be taught in the *Kaiseijo*. Professor Munroe, now of Columbia University, was the first professor of geology and mineralogy; after him we had a series of professors from Germany. On the organization of Tokyo University, geology, with the allied sciences of mineralogy and paleontology, was separated from mining and metallurgy. Civil and mechanical engineering was likewise begun in the *Kaiseijo,* and afterward formed a section in the faculty of science in Tokyo University.

Systematic meteorological observations were begun at

the suggestion of a German, Dr. Knipping, a teacher in the *Kaiseijo,* and a central meteorological observatory was established and placed under his direction. At present it is under a Japanese superintendent and staff, and is in telegraphic communication with numerous stations all over the country, including Formosa, Korea, and Manchuria. It is not strictly proper to speak of seismology as introduced from the West, for it may be said to have originated in Japan with the investigations of Professors Wagener, Milne, Gray, Ewing, Knott, Sekiya, Omori, and others; but its first investigators came from Europe, and its methods are those of the Western science.

The Department of Public Works (not now existing), being in urgent need of a large number of trained engineers to carry out its various works, opened an engineering school as early as 1871; in 1873 it invited from Great Britain a band of professors, with Dr. H. Dyer as principal, and including, among others, such men as E. Divers, J. Milne, W. E. Ayrton, J. Perry, and T. Gray. They organized an engineering college, entirely British in its character; students were dressed in a uniform, of which a Scotch cap formed a part, and were lodged and boarded in British style under a purely British management. There were sections of civil engineering, mechanical engineering, architecture, telegraphy, chemistry, and metallurgy and mining. Many of the foremost engineers of the present day are graduates of this college. In 1886 the college was incorporated with Tokyo University to form the Imperial University of Tokyo, of which, together with the engineering sections of Tokyo University, it became the College (or Faculty) of Engineering.

The first introduction of scientific agriculture must be attributed to General Capron, chief of the Agricultural Bureau of the United States, who came to Japan in 1871 as

adviser to the Hokkaido (Yezo) Colonization Bureau. At his suggestion an agricultural college was established in Sapporo with a staff of American instructors to train men to become leaders in the work of the colonization of Hokkaido; several students were also sent to America, and it is to be noted that among these students were several young girls, the first sent abroad by the government (Princess Oyama, Baroness Uriu, Miss Tsuda, among others). Hokkaido, and in particular the Agricultural College, was thus very much under American influence at the start, and retains to this day traces of that influence (the present director of the college was its former pupil and afterward a graduate of Johns Hopkins). The college, however, has lately come under German influence, which, as already remarked, has been predominant in the domain of higher learning during the last two decades or more; it now forms a part of the Northwestern Imperial University as its college of agriculture. In the meantime an agricultural school was opened in Tokyo as early as 1877, and a school of forestry in 1881; the two schools were amalgamated in 1886 to form a college, which again became a part of the Imperial University of Tokyo in 1890, and has at present five sections of agriculture, agricultural chemistry, forestry, veterinary science, and aquatic products. This college was from the first under German influence, several of its first professors having been Germans.

In literature we have always had an American or an English professor of English literature, from the days of the old *Kaiseijo* soon after the Restoration down to the present day, in the Imperial University of Tokyo, besides instructors in the English language. So also there have been a German professor of German literature and a French professor of French literature, although these chairs were not established

until a much later date. Of course, Japanese and Chinese literatures have always formed a part of the curriculum of the university, and I should not mention them here, for they do not come under the category of Western learning, but for the remarkable fact—which well illustrates the spirit that actuated the university authorities of those days—that about 1887 an Englishman, Professor B. H. Chamberlain, was for a time appointed to lecture on philology and *Japanese* literature. Professor Chamberlain was, indeed, a profound Japanese scholar, but there were many Japanese who were better scholars than he; they, however, did not know the modern methods and could not give such systematic exposition as Professor Chamberlain. Lectures are also now being given in Russian literature. In the Imperial University of Kyoto lectures on English and German literatures are given by Japanese professors, as also in the private universities of *Waseda* and *Keiô*. There is a great deal of interest taken in recent works of modern European novelists and dramatists, especially of Russian and Scandinavian writers, among a section of young Japan, which no doubt will have some influence on the future intellectual life of Japan, but it seems rather doubtful whether they will seriously affect the mass of the people.

The culture of the pre-Meiji era had been founded on Chinese classics and Buddhist philosophy, and in the earlier days of the introduction of Western learning little or nothing was known of Western philosophy; but shortly before the Restoration, books on the subject began to be introduced, and for some time thereafter such works as the text-books on ethics and political economy by Dr. F. Wayland, of Brown University, were read in schools of the English language; in higher classes, Guizot and Buckle were read, while in French schools Montesquieu and Rousseau were used. In

the *Kaiseijo* logic and psychology were taught with Mill, Fowler (deductive logic), Haven's "Mental Philosophy," etc., as text-books. On Professor Toyama's (see page 706) return from America in 1876, where he had graduated at Ann Arbor, works of Bain, Jevons, and Spencer were introduced, and Professor Toyama began to lecture on Spencerian philosophy, which became very popular in Japan. Professor Fenollosa, who afterward did so much to make Japanese art known to the Western public, came out to Japan when as professor of philosophy, and introduced students to German and especially to Hegelian philosophy. About 1890 Dr. Inouye Tetsujirô came back from Germany, and by his wide reading and retentive memory has been of eminent service in introducing students to various phases of Occidental and Oriental philosophy. Lotze, Nietzsche, Schopenhauer, etc., have not been without their exponents in Japan. Experimental psychology was introduced by Professor Motora Yûjiro (a graduate of Johns Hopkins) in Tokyo and by Professor Matsumoto Matataro (a graduate of Yale) in Kyoto. Christian theology has not occupied a prominent position either in Tokyo or Kyoto Imperial Universities, although touched upon by Dr. Anesaki in Tokyo and Dr. Gulick (of Dôshisha) in their lectures on the science of religion. There are, however, several Christian colleges supported by missions or by endowments, where it is the principal subject of instruction. The Dôshisha in Kyoto, founded by Dr. Neeshima and maintained largely by endowments from America, must be specially mentioned in this connection; it has this year (1912) made a new departure in opening a college of law and economics.

Before closing this hasty and rough account of the introduction of various branches of Western learning, it is proper that I should say a few words about foreign professors.

They generally come out to Japan on a contract to serve for a term, usually of three years, which is renewed from time to time if satisfactory to both parties. Thus no small number of them have occupied their positions for fifteen, twenty, or even more than twenty-five years, so as to celebrate their silver jubilees, and have retired with a decoration from the Emperor, a pension from the government, and the title of honorary professor from the university. Very often we have had to part with a good professor because he had been offered a better and permanent position at home. On the whole, we have been fortunate in our foreign professors, the majority of them having been men of high character; and not only have they been good teachers, but many of them have made original researches while in Japan, which have won them distinction in their respective specialties. At the same time, we have sent our best graduates abroad to prosecute further studies under eminent professors in foreign universities. In earlier days more students were sent to America and England than to any other country; but for the last two decades or more most of the students from the universities have gone to Germany, that country offering the greatest facilities for the prosecution of higher postgraduate studies. They have on their return taken positions vacated by foreign professors going home or created by the development of education and learning.

We have thus traced the history of the introduction of Western learning from its beginning down to the present day. We Japanese have always been ready to take from others what we have considered to be good for us. When we came in contact with the Chinese civilization and Buddhism in ancient times, we at once introduced them and adopted Chinese literature and Chinese and Buddhist philosophy as our own, and they have formed the main subjects of culture of our scholars. Our administrative system and

laws were modeled after the Chinese, although they were afterward greatly modified so as to become better suited to our own needs. So when we first came into contact with Europeans in the sixteenth century, we welcomed them and were eager to receive instruction in what they had to teach us. Christianity, likewise, was at first well received not only by the people, but also by men of authority and influence, until they perceived that behind it there was a great danger to the country. Even then they were desirous of keeping the advantages of foreign intercourse, if only they could at the same time keep out the dangers of Christianity; and it was only when they found that this was impossible that they had recourse to the extreme step of prohibiting foreign intercourse almost entirely. But while stringent measures continued to be taken against Christianity, the desire for new knowledge gradually became too strong to be resisted; the spirit that animated Maeno and his fellows in their efforts to read the "Tafel Anatomia" in their earnest search for truth is the spirit that has always animated the best element of intellectual men of Japan. This spirit, kept up in the incessant and untiring struggles of the Dutch scholars to introduce new knowledge among their countrymen under the shogunate, has blossomed forth under the wise policy of the open door explicitly enunciated in the fifth article of the memorable oath of the great Emperor Meiji, and under the sunshine of encouragement given to education and learning during his long and glorious reign. We flatter ourselves that at last we have succeeded in assimilating Western knowledge, and have now entered the comity of intellectual brotherhood; so that while we shall continue to learn from the West what it has got to teach us, we shall also furnish our quota, small perhaps though it be, to the common stock of the knowledge of the world.

DAIROKU KIKUCHI.

THE STUDY OF POETRY[1]

I

THE FUNCTION OF A UNIVERSITY

THE inauguration of a new institution of university rank is a fitting occasion for reviewing the field which such institutions exist in order to cover; for going back for a moment to first principles, and endeavouring to state, in the simplest terms, why such institutions exist, and what they may effect towards the moulding of a new generation, and the elevation of civic and national life. Different universities, according to the circumstances of their foundation and history, can shew different reasons for their existence and for being such as they are. But all of them, whatever the date of their origin, whatever the place of their settlement, have come into being in response to certain demands of the place and the time. All of them have been founded with a purpose single in its nature, though diverse in its manifestations. That purpose is to make stated and secured provision for the higher needs of a civilised community. The needs, like the pursuits, of a community are many. But its civilisation is one. It is the object of a university to gather up that civilisation, to analyse and study its separate elements in order to recombine them at a higher power, and thus to give conscious direction to the human mind in its knowledge of the past, its understanding of the present, and its power over the future. Its office is to store up, to sort out, and to impart knowledge; and by doing so it accumulates, organises,

[1] A discourse prepared for the inauguration of the Rice Institute, by Professor John William Mackail, formerly Professor of Poetry in Oxford University.

J. W. Mackail

and gives forth power. Knowledge is power, according to the old saying; it is latent or stored power. Conversely, power is knowledge transformed into energy; knowledge in action. Education, the process which goes on in a concentrated form and at high pressure in a university, is a mechanism by which the potential energy of the human mind is developed, disciplined, cleared for action. Knowledge is indeed an end in itself, and one which has a value that may properly be called inestimable, since it cannot be expressed in the terms of any other value. Riches, comfort, health, fame, influence, beneficence, are things of which knowledge pursued for its own sake and as an end in itself takes no heed. But while knowledge is or may be an end, education is only a means: a means to knowledge, for such as desire knowledge for its own sake; but for all who pass through it and undergo its influence, a means to the practice and conduct of life.

Hence in any community the idea of a university, the sort of education which a university will be planned and meant to give, will depend on the kind of life which that community desires, aspires towards, sets before itself as worth attaining. In the ancient world the earliest institutions to which the name can in any sense be applied were religious colleges —schools of the prophets, as they are called in the Old Testament, or training-colleges of the priests, as they existed and flourished in Babylonia or in Egypt. The knowledge and power after which they sought, which they accumulated, recorded, and transmitted, were the knowledge of and the power over supernatural forces. For these supernatural forces were then, according to the common belief, what governed the life of mankind and held it at their mercy; to understand them and their ways, and thus to gain the power of foreseeing their action, propitiating their favour, giving

this or that direction to their working, was no mere matter of abstract study or idle speculation: it was most severely and immediately practical; it lay at the root of individual and national prosperity. Without education in this all-important and all-embracing knowledge, industry and commerce, arts and manufactures, the conduct of war and peace, were blind and helpless: in a word, life was impossible.

Out of that world rose, after many ages, what we know as the classical civilisation. This was the work of Greece, carried on and consolidated by Rome. The universities of the Græco-Roman world belong to the same period which saw, for the first time, the rise of a trained governing class of organisers and administrators. Hence in these universities the subjects pre-eminently studied were those necessary for such a class: oratory, law, politics, and finance. At the same time the creation of a trained governing class set free those who did not belong to it, whether excluded by birth and fortune or holding aloof by choice from active pursuit of the duties attaching to the work of government. These, and especially the latter class, those who deliberately abstained from active public life, might now pursue knowledge for its own sake; and other universities arose which, in response to this new demand, devoted themselves to the sciences: on the one hand, to the pure or abstract sciences, those of the human mind, like grammar and logic and metaphysics, and those of the physical world, like botany or chemistry or astronomy; and, on the other hand, to the applied sciences, such as engineering or mechanics or medicine, or to those sciences which are also arts, like rhetoric or music.

When, in the Middle Ages, men began to gather together the wreckage of the ancient civilisation and to reorganise life on a fresh basis, their notion of a university was fun-

damentally different. For the mediæval notion of the world was that it was something limited, precise, and ascertainable. It was something of which complete knowledge was possible; and to give this complete rounded knowledge was the function of education. The forms of life were prescribed by dogma; and the substance of life, on all its sides and in all its manifestations, was what could be comprehended in these forms. Just as theology was fixed and bounded by the authoritative doctrines of a universal or Catholic Church; just as political and social life was fixed and bounded by the equally authoritative constitutions of the universal Roman Empire, which held sway over men's minds long after it had itself ceased to exist except as a memory of the past or a dream of the future: in like manner and to a like degree were the form and the content of all knowledge determined and limited. Treatises were written *de omni scibili,* 'concerning everything which is capable of being known.' This was an ideal, in so far as few, if any, had the vigour of mind, the industry, acuteness, and patience, that were required for its attainment. But it was, given these qualities in the student, an attainable ideal. A university professed to offer, its students came prepared to receive, universal knowledge. The mediæval curriculum—the *trivium* and *quadrivium* of educational legislators—was the same everywhere; was one, complete, and unalterable. Study might be pursued further in certain branches of it than in others; but that was only in so far as the student failed to complete the full course which would leave nothing more to learn or to know. The *Summa Theologiæ,* the sum and substance, over all its range and into all its details, of divine knowledge, was actually put together and written down; the *Summa Scientiæ,* the sum and substance of secular knowledge, was the under side, as it were, of that other fabric, and could not extend beyond its limits.

That is to say, all learning, both liberal and technical, was the province of a university; the scope and limits of all learning were imposed from without by a dogmatic and omnipotent theology, and whatever knowledge lay beyond these limits was either proscribed as sinful, or its existence was denied.

Hence the human mind was not only bounded but crippled. Practice did not, indeed, follow theory to its rigid consequences. Schools of medicine, of civil law, even of natural science, grew up here and there, and flourished precariously under the jealous eyes of the Church. Art grew up of itself, without any systematic art-training. Architecture and engineering were in the hands of guilds, where knowledge was transmitted, in theory and practice, as a secret treasure from father to son or from master to apprentice. Painting and the sister arts wrought out a tradition of their own. Poetry insisted on making itself heard, but was discountenanced as heathen vanity or worse. The brilliant culture of Provence, which had gone out of Europe to the Arabs for a new and larger life, was crushed by armed force, and perished under the sword of the so-called Crusaders or in the fires of the Inquisition. Even at the end of the Middle Ages, and when the new world of the Renaissance was forming itself, Chaucer, the first of our own poets, ended by a formal and express disavowal of his own poetry, revoking and retracting it (all except legends of saints, and homilies, and books of morality and devotion) as vanity and sin. Physical science was equally suspect, and was subject, down to the time of Galileo and later, to equally jealous control and equally vindictive persecution.

This tyranny of theology lasted long enough to affect the modern universities likewise, down to a time which is within living memory. It was not broken either by the Reforma-

tion or by what is called the Revival of Learning. For the Reformation, as indeed its name implies if we consider its real meaning, only recast that tyranny in a new shape, 'reformed' it and imposed it afresh on the human mind; and the Revival of Learning was a partial, imperfect and agonising struggle to regain that freedom of the intelligence on which all freedom and all progress ultimately depend. The pre-Revolutionary foundations in the American Colonies, like Harvard (the mother and head of American universities) and Yale (created in the first year of that eighteenth century which was the liberating age of human thought), were theological colleges, restricted by the tenets of Puritanism, and regarding all kinds of secular learning as subsidiary elements towards theological training. Fifty years after the foundation of Yale the first decisive step towards the liberation of knowledge was taken. The University of Pennsylvania, founded on lines laid down by Benjamin Franklin in 1751, led the way in the English-speaking world towards the conception of a seat of learning in which learning should be unrestricted by dogma and have no limits set to it other than the limits of human intelligence and capacity. That foundation, originated by men who were to be the creators of the American Commonwealth, was an achievement in the field of human thought which marks a new epoch, just as the foundation of the Republic a generation later marks an epoch in the political and social life of mankind.

The step then gained has never been lost. More and more surely, as time went on, the declaration of intellectual independence made by those pioneers of the modern world became a profession of faith and a standard of conduct throughout the international commonwealth of learning. Progress was slow: it was not until 1871 that religious tests were removed from the ancient universities of Oxford and

Cambridge; here, as elsewhere, the creators of the United States led the way, and the American Republic followed them in the advance towards a new conception and a new conduct of life. It became, in the full sense of the words, a New World.

That world existed at first, and for long, only as a sketch or outline: it drove its outposts forward through virgin forest or over empty prairie; the advancing tide, however swift its actual advance, required generations to fill up the channels laid for it and widen out into lakes and seas; the foundations were pushed on, here and there, at random, and the earliest superstructures built upon them were often slight and mean. It was not until after the Civil War that the American nation, secure in its unity and conscious of its greatness, began to organise its own higher education, and to realise the full culture of the human faculties as a function of its national life. Since then the growth, in all the States of the Union, of institutions of liberal and technical learning has been rapid and vast. Yet even so, it has hardly kept pace with the enormous growth of population, of civic organisation, and of material resources. The new institution now being inaugurated at Houston is one among many such new foundations, and they will not be the last. The foundations are laid, but the structure towards which they are laid is only begun.

But while the number of American universities is steadily growing, the ideal of an American university is undergoing no less striking and fruitful an expansion. It is being recognised that a university, or any institution of university rank, must have a sphere of study and of influence as wide as the whole width of human activity. It can no longer confine itself to some special study; it can no longer be merely a theological seminary, or a school of letters, or a training-college

of commerce, or a collection of laboratories and workshops. Its function and scope must be universal. It must proclaim the unity of all knowledge, the kinship of the arts and sciences, the mutual interdependence of all study and research towards the conquest of nature and the complete civilisation of man. To this task there are no bounds; beyond the widening frontiers of knowledge lie ever more and more unexplored territories. To the Republic of Learning no limits are set by the ocean. The growth of knowledge is the growth of power; the organisation and communication of knowledge are the organisation and communication of power; and that power is not merely a power over what is known, but a power and a will and an endless purpose to know more.

It is, then, matter of congratulation that the founders of this institution have determined that its studies shall not be confined to any single branch of knowledge, but that the technical and professional instruction which it will offer shall be liberalised by organic connection with art and letters. In the stately and ample surroundings which have been planned, with the large and varied equipment which is being provided, the Rice Institute gives welcome promise of rising to the height of the opportunity presented to it. By a wise munificence, it will offer its education free to its students; it will lay no tax upon the acquisition of knowledge. By an equally wise breadth of view, it will base professional and technical training on a liberal general education, and will thus affirm the human side of science, commerce, and industry, no less than the scientific, commercial, and industrial value of art and letters—of "the so-called humanities," to quote a phrase from the authoritative statement of its Governing Board, which derive that name from a recognition of the fact that human life, at its broadest and fullest, is the

subject-matter alike of all academic study and of all civic endeavour. It is proposed to assign no upper limit to the educational activity of the Institute; nor, indeed, is it right that any such limit should be fixed except that fixed by Nature herself—the limit of human activity and capacity. But its upward growth will be on broad foundations; its roots will draw life from a large and rich soil; and the hope may be expressed that its lateral radiating growth will, no less than its upward growth, be subject to no imposed limit. For only thus can its full natural expansion be achieved and its organic vitality secured.

Among the "humanities"—among those studies or pursuits in which the noblest instincts of human nature are fostered and its highest aspirations sustained—poetry takes a high, if not the highest, place. As language is the universal and necessary instrument of thought, and as thought is the source and motive power of all action, invention, and creation, so poetry is the organ of language and thought at their highest power, in their most intense and most vitalising manifestation. It will not, therefore, be irrelevant to the inception of a new university to consider more closely, first, what poetry is, and then—a matter of no less moment and of a practical importance which will appear in the development of the discussion—what is the task or function of poetry in the modern world.

II

WHAT IS POETRY?

IN order to discuss anything rationally, we must first have a clear notion of what the thing is which we are discussing. Most misunderstandings, most false opinions, arise from mere confusion; and the heat of debate increases with the vagueness of definition. Even in the sphere of the physical sciences, where perpetual reference back to facts is implied in the nature of the case, and where these facts are visible, tangible, and ponderable, such confusion is not unknown. But the confusion is more apt to arise, and can spread further without detection, in matters where theory cannot be so readily, and has not to be so constantly, brought to the test of experience; where experience itself is fluctuating, and subject to the distorting influence not merely, as in physical science, of tradition and habit, but also of unreasoned instinct and variable emotion. Only by the continuous effort of generations have the physical sciences been brought into the state in which their really scientific pursuit is secured; only by constantly applying them to practical problems can we test the truth of generalisations and the relevance of theories, and be sure that our knowledge is real knowledge, and bears relation—a real and helpful relation —to the actual world in which we find ourselves and with which we have to deal.

In what are called the humane studies—those of art and letters—the same twofold necessity exists: the necessity of a clear definition of terms, and the necessity of testing the value of any study or pursuit by laying it alongside of facts

and seeing what relation it bears to the claims of life. Before considering, as it is my main object to do, the function and task of poetry in the actual modern world, whether as a subject of study, an art in practice, or (more largely) an element in civilisation, it will be proper, and indeed necessary, to clear the ground by saying what poetry is.

In this as in so many other matters the instinctive tendency in many minds is to give to the question, 'What is poetry?' the answer, 'I know, so long as you do not ask me.' And it is no doubt true that most people have some vague and general conception of what is meant by the word 'poetry' floating in their minds. But their conception is so vague and indeterminate as to be of little use. That poetry is a kind of language, differing in its nature alike from the ordinary language of our daily intercourse and from the language used in books of science or philosophy or history, of treatises on politics or economics or religion, of memoirs or essays or narratives, would be generally admitted; but when we go beyond this and ask what is its specific nature, many would be unable to say more than that it is language arranged in lines of a certain arbitrary length, with the words so artificially ordered as to produce an agreeable effect upon the ear, and to excite a certain pleasure, comparable to that produced by music, in the senses of the reader. Beyond that, they would have to fall back on instances: poetry, they would say, is, in ancient literature, Homer and Virgil; in our common English inheritance, Chaucer, Spenser, Milton; in more modern times, Wordsworth, Keats, Tennyson, Browning on one side of the Atlantic, Bryant, Longfellow, Whittier, and a thousand other writers who have succeeded to them, in our own Republic.

But what are we to think of these and other poets, not merely as men, not merely as writers, but as poets? What

is that thing called 'poetry' which they all produced, and what are we to think of it, as an art, as a way of occupying life and affecting the lives of others, as a subject to be studied or a craft to be exercised? When we come to this point we are faced at once with the confusion which arises from the absence of a clear notion of what is meant by poetry, and from the consequent absence of any firm common ground when we try to state and to appraise its function, its value, its relation to the task, the duty, the privilege of actual men and women here in the twentieth century. This confusion affects the eulogists and the detractors of poetry alike. Many wild words are spoken on both sides. It is needless to enlarge on this notorious fact. On the one side are the devotees of poetry, who regard it as something too lofty and sacred for definition, as something that stands outside of and apart from common people and their pursuits. On the other, in much larger numbers, are those who think of it as a rather trifling amusement, suitable for people who have nothing better to do; or even as something vicious and demoralising, something that weakens the mind, destroys industry and accuracy, cultivates fancy and sentiment at the expense of intelligence, and is a stumbling-block in the way of the pursuit of truth. To them poetry is like alcoholic liquor, a dangerous servant and a destructive master. 'One of the Fathers,' says Bacon in his 'Essay of Truth,' 'called Poesy *vinum dæmonum* (devils' wine), because it filleth the imagination with the shadow of a lie.' The churches, and religious people generally, have always, if they did not go as far as St. Augustine, at all events regarded poetry with suspicion, and not been comfortable about it. And here they are, for once, in agreement with the rough common-sense of business men who care for religion as little as they care for poetry. It is easy to laugh at the mathematician who

asked of Milton's 'Paradise Lost,' 'What does it prove?' But it is not so easy to ignore the man in the street who asks of poetry, not 'What does it prove?' but 'What sense is there in it?' It is not so easy to confute, before a careless public, the discontented man of letters who turns against his own art, and says of poetry, in the words of a contemporary of Shakespeare, that it is a thing 'whereof there is no use in a man's whole life but to describe discontented thoughts and youthful desire.' To such minds poetry is either a childish folly or a deliberate misapplication of human powers.

Against such an attitude we may set the many splendid tributes in which, while pretending to give a definition of poetry, the poets themselves have claimed for it qualities so marvellous, a value so great, that nothing else in life is so precious. Wordsworth calls poetry 'the breath and finer spirit of all knowledge.' Shelley calls it 'the record of the best and happiest moments of the happiest and best minds.' Matthew Arnold says that it is not only 'the most perfect speech of man,' but also 'that in which he comes nearest to the truth.' When poets commend poetry, their testimony may be taken by the outer world with some of the suspicion which attaches to people who cry up their own wares. Yet even after making all due allowance for this, the two attitudes of mind towards poetry are clearly inconsistent with each other. We may admit that there is truth in both, as there is truth somewhere at the basis of any widely and sincerely held opinion on matters which affect life. But if both are true, they are clearly not both true of the same thing and in the same sense. In order to reconcile them in any wider and more comprehensive truth, we must try to avoid on the one hand the glitter of rhetoric and sentiment, the 'luminous mist' (in Coleridge's fine phrase) which imaginative artists are apt to wrap round their own art, and on the other hand

the impatience of the practical and unimaginative man with anything that falls beyond the scope of his own daily experience, that uses terms with which he is not familiar and aims at objects which he has not learned to appreciate. And the best way towards arriving at common ground is to define our terms as clearly and simply as possible.

With this object, let us now proceed, not to praise or blame poetry (both are easy, and both are useless), but to explain what poetry is. I will first state the technical definition of poetry; from it, and keeping it in view, we shall be able to frame a substantial or vital definition of it, to define it not merely as a technical term, but as an organic process or function. Like all other arts, poetry has both sides. Like music, painting, or architecture, it is a thing subject to laws which can be taught and learned, historically studied and practically applied. Like them, it is also not merely an art, but a fine art; that is to say, it is a form of creative human activity, bearing an intimate relation with the energies of human nature, and with the outlook of man upon the material and spiritual world.

Poetry is, formally and technically, patterned language. This is its bare and irreducible definition. Its specific quality, its *differentia* from other kinds of artistry exercised on the material of language, is that it works language into patterns and uses it not only for its common and universal purpose of expressing meaning,—not only for its heightened or artistic purpose of expressing meaning in such a way as to express it beautifully and thus satisfy the artistic sense,—but also, and expressly, so as to bring it within the scope, and make it subject to the laws, of that kind of decorative designing which we call pattern.

Some brief further explanation may here be added to make the point quite clear. When we are defining poetry

and separating it formally from other kinds of spoken or written language, it is not enough to say that it is language which possesses design and has decorative value. All beautiful, dignified, and elevated language has that. The quality peculiar to poetry is something different. We may call it, as we choose, a decorative or a structural quality: for what lies at the root of all true art is, that in it structure and decoration are inseparable; each implies the other, and each exists, in any artistic sense, only by virtue of its essential relation to the other. Structure in the abstract, apart from the decorative quality through which it manifests itself to the senses and affects the imagination and the emotions, is matter of science, not of art. Decoration in the abstract, apart from the material in which it is wrought and its relation to the structure which it decorates, is meaningless. The synthesis of the two constitutes beauty; their vital union is the aim of art. Now the specific quality of poetry as distinguished from other kinds of literature is that in construction and decoration (its construction being decorative, and its decoration constructive) it follows the laws of pattern. The essence of pattern, as is well known to all pattern-designers, consists in its having what they term a repeat. Pattern is built up out of, or grows out of, a repeated unit; and the art and skill of the pattern-designer are shewn by his success not merely in making the repeat mechanical, but in so handling it that the whole field over which it extends has a beauty and a unity of its own, rising out of and yet distinct from the quality of the repeated unit. A row of equal dots is a pattern in its crudest and simplest form; these dots may be grouped, and the group repeated; these repeated groups may be themselves regrouped into a larger design, and that repeated; and so on. Not only so, but when the pattern is to be executed by hand and not by a machine, it may be treated

flexibly and varied subtly; it may depart from exact repetition without ceasing to be a pattern so long as the repeat, or its main elements, continue to be felt. All really excellent patterns, patterns which are works of art, have something of this flexibility. It may extend so far that the repeat has to be sought for, is visible only to the trained eye, and affects other eyes with a pleasure which they feel but cannot analyse and do not fully understand.

This is well understood as regards the arts of painting and music. It is less well understood as regards the art of poetry; but it is true of poetry equally with the other arts of pattern. Poetry, according to a definition which in all probability comes to us from no less an authority than Milton, is the kind of language which 'consists of rhythm in verses.' Prose also has rhythm, and its rhythm may be of great and intricate beauty, but it is not 'in verses'; its rhythm is not subject to the law of repeat. It is indeed the essence of prose that it has not a repeat; so much so that when it slips into a repeat it becomes bad prose, and affects us disagreeably. This is what its name means: 'prose'—the Latin *prosa oratio*—means language which moves straight forward without a repeat in its rhythm. Similarly, 'verse' (also a Latin word) simply means repeat.

The distinction then between prose and verse is fundamental. It is not quite the same as the distinction between prose and poetry; for while no prose is poetry (except in a very loose and figurative way of speaking, unhappily not seldom used), all verse is not poetry. All patterned language is verse, but to make it poetry the pattern must be skilfully designed and governed by the sense of beauty. Or, if we like, we may say that poetry and verse are the same, only then we must include bad poetry as well as good. It is simpler to say that bad poetry is not poetry at all. Milton again

here supplies us with an illuminating phrase. In the 'Paradise Lost' he speaks of 'prose or numerous verse.' Verse which is 'numerous,' in which the repeated unit and the way in which the repeat is managed are alike beautiful, is poetry.

The scope of pattern, in language as in all the other materials upon which human craftsmanship is exercised, is very wide. Its development varies from country to country, from age to age, from one school of artists to another; and even the same artist may use it very variously at different times and for different purposes. It suffers alternations of growth and decay: a period of healthy growth is succeeded by one of stagnation and disintegration, out of which again in time fresh growth arises. The condition of decorative art in any nation is, at any time, an index to the state of its civilisation; for art is a function of, or an element in, the whole process of national life. Art in a sense exists for its own sake; but in a more important sense it exists for the sake of the human life in which it is a factor. Just as, amid great varieties and fluctuations of movement, there are traceable certain broad lines of national development, so it is with the decorative arts of a nation, and with poetry among these: there are certain normal or dominant types of pattern; on these each artist varies according to his own imagination and skill; and from the normal and central type extend outwards in all directions other types, continually in process of invention, cultivation, and change. Some of them are experiments which come to nothing; others strike root and become important enough to affect or alter the normal type of pattern. Thus the art of poetry is always renewing itself through fresh invention under the stimulus of individual genius, and always rebalancing itself through a slow but final current of judgment as to the success or failure of the new type. Instances may be found anywhere by even a cursory

glance over contemporary poetry. But we shall be on clearer ground if we put aside living authors and look to the work of an earlier generation, which has already taken its place and can be looked at as a whole and from a distance. Among American poets of the last century we shall find the normal patterns of language, for instance, in the work of Longfellow, perhaps still the greatest, as he is the sweetest-voiced and sanest-minded, of them all. Notable divergences from normal pattern may be seen on the one hand in the lyrics of Poe, with whom curiousness of pattern was almost an obsession; on the other hand, in the singular and hitherto unique work of Walt Whitman, in which the reaction against formalism of pattern went so far that it has been questioned whether any pattern, in the strict sense, is left at all: or in other words, whether the contents of 'Leaves of Grass' are, or are not, poetry.

Poetry, then, according to its formal and technical definition, is patterned language, the material of words wrought by art into patterns; and it gives the pleasure, partly sensuous and partly intellectual, which all pattern gives through, and in proportion to, its decorative fitness and beauty. If we regard it not on its technical side, but in its substance and meaning, it has a corresponding definition: it is the art or process which makes patterns out of the subject-matter of language. That subject-matter is life.

As soon as we have grasped this truth firmly we shall understand the things which the poets have said about poetry. Life, as it presents itself to us as we pass through it, has no pattern, or at least none (except to some people of very simple and fervid religious belief) that is certain and intelligible. It is multiplex and bewildering; its laws are confused; it does not satisfy our hopes or our aspirations: sometimes it seems purposeless, often it seems, as Hamlet

says, 'out of joint.' It makes no pattern; still less does it make a pattern of beauty. The high office, the unique function, of poetry is to compose this disorder into a pattern; to bring out, make visible, lift up as a light in darkness, the particular portion or aspect of life which it touches; and in the hands of the greatest poets, to do this with life as a whole. In the beautiful words of Shelley, which I may now quote with the hope that their significance can be understood, poetry 'makes familiar things be as if they were not familiar.' It shews us the confused, depressing texture of experience in a new and strange light under which we can realise it as part of the divine order. It lets us see life in its inherent beauty and value, and gives us strength to live.

Thus poetry is, in no mere rhetorical or sentimental sense, the highest human achievement. It is the culminating point of that wide combined effort or instinct which is at the base of all education, of all study, of all work; and this is, to realise the potentialities of life, to master the world and enter into our inheritance. To do this is, in the full sense, to live.

III

THE MODERN WORLD

THE present age is in a state of rapid flux. Not in one country only, nor among one social class only, but everywhere from top to bottom and from end to end, change is proceeding with unexampled speed. All movement, not only physical but intellectual and moral, has been vastly accelerated. The old barriers are everywhere breaking down, the old ideas and organisations disappearing, or in course of being fundamentally transformed. An enormous stock of hitherto latent energy has been called violently into action, and to this process it is not yet possible to assign any limit. We live, and our children will live after us, among the wreckage of an old order, and the girders and scaffolding of a new one which is arising, amid dust, confusion, and seeming absence of any mastering control or intelligible design, to replace the old.

The nineteenth century, which now lies so far behind us that we can more or less look back upon it as on a past age and receive from it a general total impression, was an age of ideas, and of belief in ideas. Among its dominant ideas were those of nationality and of enfranchisement in politics, of organic continuity in history, of conquest of the physical world in science. Such ideas, grasped, believed in, and practically applied, impressed upon the century a character of its own, and one wholly different from that of any previous age. They were all summed up and included, together with many others of hardly less significance, in the governing idea of progress. Progress was necessarily accompanied by change;

but change was sought not for its own sake, but for the sake of giving effect to the ideas which lay behind it as a motive force. Change was realised as development: this was the achievement of science. Development was assumed to be progress, and was hailed as such: this was the essence of liberalism. It was an age of unbounded hope for the future and of active belief in the work of the present.

A generation ago, a change began to pass over the human spirit. The reflex action of the new ideas cut them away from the base out of which they had sprung. For ideas, like other things, are subject to the law of development, and pass through an orbit of their own. The revolution of the nineteenth century has, like other revolutions, 'devoured its own children.' Its ideas have partly dwindled, partly failed, partly so altered and expanded that they can no longer be recognised for what they were. The law of development has, in the phrase of engineers, 'taken charge.' In discovering it, we have discovered our master. It is a law not of our making, and but little under our control. Before its march all the old traditions, and all the moral or customary sanctions which attached themselves to these, crumble away or go off in smoke. It is a power not only invincible, but incalculable. We may still talk of progress; but many of us have in our hearts ceased to believe in it; or if we do believe in it, it is a different thing in which we believe from that progress which quickened the impulses and inspired the actions of our predecessors. Progress meant to them betterment. It meant the coming of mankind, with certainty and with increasing rapidity, into their inheritance; and that inheritance was assumed, or believed, or as men thought, proved, to be a goodly inheritance, to include in it all good. The inheritance which we now see lying before us seems rather a burden than an enfranchisement. It is an 'importunate and heavy

load.' Long ago, the greatest of the Hebrew prophets cried out sorrowfully to the Power which ruled above, 'Thou hast multiplied the people, and hast not increased the joy.' Some such feeling now weighs upon the present age. The Power goes on its own inflexible, sinister way, and forces us on before it. We find it more and more difficult to believe that it works for good; for we do not see it doing so. There is a wide-spread belief that progress, in the old sense of the word, does not exist.

The denial of progress, as a ruling law of life, has also been a doctrine held in past ages. But they differed from the present age in this, that they carried out their doctrine in practice. They were conservative. They tried, with all the power they had, to fix things as they were, lest a worse thing should come upon them. This was the whole effort of the Middle Ages. It was the effort of the conservative or reactionary element in society which strove, persistently but in the end helplessly, against the intellectual revolution of the eighteenth century, the industrial revolution which succeeded it, and the political, scientific, and social revolutions which have carried on the process down to our own day. But conservatism in the old sense has also ceased to exist as a real and effective doctrine. Change has been realised as an invincible force; the desire for change has become a fixed instinct; and to this, rather than to any reasoned belief or any assured hope, is due the intense restlessness of the modern world.

The solvent effect of many forces has co-operated to bring this state of things about. Intercommunication in space has reached such a pitch of ease and regularity that the communities of mankind are no longer cut off from one another; what affects one, almost at once begins to affect all, and an impulse towards change arising anywhere from fresh

ideas or altered circumstances is propagated, as it were by waves travelling in all directions through an elastic medium, over the whole world. An immensely increased knowledge of the past has come to men from the compilation of records and the organisation of research; and the historical method (perhaps the greatest single invention of modern times) has interconnected all that knowledge so as to make it breed and multiply through mutual fertilisation. Knowledge and understanding of so many past changes has brought about an attitude of mind in which nothing is seen to be unchangeable, in which no change seems impossible, in which life itself appears to consist of change. The development of applied science and the triumph of machinery have opened up a boundless prospect of the degree to which this inherent law of change may be utilised, may be turned by mankind to planned ends and foreseen purposes. Together with all these solvent influences is to be reckoned another, negative indeed, yet in its effect perhaps the most potent of all. This is the disappearance of religion, in the older and original sense of the word. For religion as it was understood in earlier ages was a system of enactments and prohibitions based on undefined fear and sanctioned by terrible penalties; once established, it was the strongest of all conservative forces, because exercising the highest and widest controlling power over the thought as well as the actions of men.

The joint result of all these solvent influences in their accumulated force is a movement of change so rapid and so wide-spread that all the old framework of life tends to disappear, and no pattern of life is left. The course of change points everywhere, which is the same thing as pointing nowhere. The compasses by which life was directed have been demagnetised. It is an age of perplexity, an age of disillusionment. This is not like the old clearing up of thought

(the *Aufklärung* of philosophic historians) which sought to dispel illusions that had gathered round and blurred a framework of certainty. It is disillusionment in another sense. Its light is a blind and formless glare in which all objects disappear. It issues in the feeling that what is to be discovered is infinite and cannot be discovered fully; that what is to be done is infinite and cannot be done effectively.

Against this relapse into chaos what is needed is a steadying influence; and this influence, while it may arise from different sources and act along many channels, is to be sought and found nowhere with more clearness and certainty than in poetry. For it is the function of poetry, as we have seen, to make patterns out of life; to discover by its imaginative vision, to make manifest by its creative and constructive power, the order and beauty, the truth and law, that underlie the flux of things. To the paralysing sense of disillusionment it opposes a revelation of essential truth; beneath the chaotic surface of life it apprehends ordered beauty. It re-creates the fabric of life; it renews the meaning and the motive of living.

It would be needless, in speaking to any educated audience, to multiply instances in which this function has actually been performed by great poets, or to point out how their quickening and reconstituting influence is not confined to their own fellow-countrymen in their own age, but retains or may even increase its effect in distant ages and among other civilisations. All the great poets of the past derive their greatness for us in the present from the fact that their effective force on life still survives. The religious poetry of the ancient Hebrew people, translated into other tongues and reinterpreted by new minds, remains a dominant power not merely among the wide-spread colonies of their own descendants, but among all the nations who have received

it as part of the inheritance of Christianity. Homer, the poet of poets who wrote the Iliad and the Odyssey, was the teacher and in some sense the creator (so the Greeks themselves claimed) of ancient Greece; but after ancient Greece had perished, and ever since, down to the present day, he has remained a powerful influence over the ideals, and thus over the conduct and action, of successive generations of mankind. Virgil, the prophet and interpreter through his poems of the Latin race and the Roman Empire, shares with the Roman statesmen, jurists, and administrators the glory of having formed and transmitted to posterity the plan of an ordered civilisation reigning throughout an organised empire and imposing itself on the outer surrounding world. The great poets of England and the English-speaking nations have, on one side or another, achieved a task hardly less. Chaucer interpreted and summed up the expansion given to life by the earlier Renaissance; he initiated modern England. Spenser gave voice to the ideals and inspired the action of the Elizabethan age. Milton engraved upon the minds of his countrymen (and among those countrymen were the Fathers of the American Republic) the doctrine, the belief, the law of conduct, which were the strength of Puritanism and the basis of Republicanism. In more recent times the poetry of Byron and Shelley carried on the work and enforced the ideas which, through the French Revolution and the movement of which the French Revolution was the symbol and centre, transformed the civilisation and life of Europe. The Brownings became, a generation later, the interpreters of that Liberalism which, in the social, political, and industrial world, was the chief motive force of the nineteenth century. In the middle years of that same century the group of American poets among whom Longfellow, Whittier, and Lowell are the most distinguished names, exercised

the most powerful influence over national life, and share with Lincoln and Grant, with soldiers and statesmen and men of action, the glory of creating and sustaining that faith and that resolution among the people which saved the Union and established a free and indissoluble Commonwealth.

Poets have not ceased; and there may be poets now alive whose work in the judgment of future generations will be comparable in the history of the world to that of their great predecessors. Whether this be so or not, the task and function of poetry remain the same; and thus the study of poetry remains an essential part of human culture, and its practice an essential element in human activity.

Among the great poets, as among all great artists, there is very wide differentiation of function. While they all, in virtue of being poets, create or embody patterns of life, these patterns are never twice the same; they are the creation of individual genius working on material which, being coextensive with life itself, is of infinite complexity and variety. In the phrase of St. Paul, 'there are divers interpretations, but one spirit.' The interpretation is never twice the same; the material to be interpreted never presents itself to two artists alike. Hence the task of poetry is never completed; it is a concurrent and endless integration of the meaning of life; and while the poetry of the past is our priceless inheritance, the poetry of the present is our ceaseless need. Some poets have been, primarily and distinctively, prophets of the future; with others, their work has been to reillumine the past and make it alive to us, to make it an effective part of our own conscious life. Others, again, have brought form and beauty into the present, and shewn us the pattern in the things that lie nearest to us. Thus Tennyson owed his vast influence and popularity to the fact that he was always just abreast of his time; he was the voice, during the

sixty years of his poetical production, of the actual spirit of his country, the thought and emotion and work of his age. Other poets as great have failed to obtain the same universal acceptance, because the patterns of life they created were of a life somewhat further apart from common experience: such poets may have to wait for their fame until after death, or may exercise their influence not so much on the world of their own time directly as on a smaller number of minds whom they inspire and fertilise, and through whom they become powerful germinal influences on a later generation. To elucidate and appreciate this complex stream of creation and its effect upon mankind is part of the study of poetry: but more than that, it is part of the study of civilisation, part of the equipment required for understanding the world and being able to deal with it, to master its springs and to sway its course.

The state of flux which I began by noting as characteristic of this early twentieth century is perhaps nowhere so marked and so rapid as in the United States. From its beginnings, and now as much as ever, the American Republic has been the laboratory and testing-ground of the whole world. The founders of the Republic set themselves to make that continent to which the name of the New World had been applied since its discovery and colonisation, a new world in the full sense; and this has remained more or less, in principle at least, the guiding doctrine of their successors. But in the framing of a pattern for this new world, poetry and the poets (except, as I mentioned, in the course of the great struggle which established the freedom and unity of the nation) have borne little part. The creators of the United States were neither poets nor much influenced in their thought and action by poetry. Washington, Franklin, Hamilton, all had a certain amount of imaginative or creative

genius; but they had minds of the prosaic, not the poetical order. The poetry of Puritanism had, a century before their time, put forth its first and last flower in Milton; unless we say that, half a century later still, the thin and austere but exquisite poetry of Emerson was a last autumnal flowerage from the same stem. There are many modern American poets, but no one among them has been recognised by the world as belonging to the first rank, or appears to be a moulding and formative influence over the national life. Of the two names whom many would hold to be the foremost among American poets, Poe was a stray exotic, and Whitman a splendid anomaly. Perhaps the national life is more confused, certainly the national history is poorer, through the comparative absence of poetry—of a national and great poetry—as one of its constructive and enriching elements. And in the solution of the vast problems which to-day confront the Republic, those patterns of life given by the poets, whether native or foreign, cannot be neglected without grave loss. It is necessary to maintain, it is at once a privilege and a duty to urge, the study of poetry as a part of the public provision for the education of the people.

This new Institute, like most modern foundations for promoting higher education, devotes itself largely or mainly to technology and science. This is quite right; for these are studies of immediate utility and pressing importance. But did it confine itself to these, it would contract its own scope and diminish its own value. Technical processes are means and not ends in life; physical science itself is based ultimately on ideas: letters and art give it not merely its interpretation, but its impulse and inner meaning. Thus the study of the humanities is at once the basis and the crown of the study of the sciences; or rather, we may say, supplies these sciences with a motive and an informing spirit.

The humanities, the studies which deal directly with the vital and human elements in life, with thought, emotion, and imagination, culminate in poetry; and we may now proceed to consider somewhat more closely and more in detail the function and task of poetry in relation to actual life at the present day. The modern world, as I said, is in a state of rapid flux and transformation. Among a thousand elements or forces which go to make its movement, one or another may be singled out as of special prominence. But there would be general agreement with any one who called the present age eminently an age of the extension and dominance of science; or who called it, no less eminently, an age of business conducted on a vast scale, at high tension, with exceeding complexity; or who, once more, called it the age of expanding and triumphant democracy. Let us proceed to regard the function of poetry in relation to these three great distinguishing features of the actual world.

IV

POETRY AND SCIENCE

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SCIENCE, as the term is now understood and as the study denoted by the word is now pursued, is a birth of the modern world. Its growth was coincident with the earlier development of the United States, where its practical application has expanded to keep pace with the ever increasing demands of a national growth more swift and complex than has elsewhere been known. Within the last two or three generations it has also taken its place as an important, and even an indispensable, part of higher education. Technical institutes have sprung up on all sides in response to public demand. The study of science has been taken up by the older universities, and is the main pursuit in most universities of modern foundation. Even higher claims are made for it. Its exponents speak of it not only as having won an assured place in the front rank of human studies, but as occupying in that rank a predominant, if not, as some of them venture to assert, a practically exclusive place. A note of triumph is sounded in these utterances. The Royal Society of England has this year [1913] been celebrating, with splendid ceremonies and before audiences containing many of the foremost names of the present age, the two hundred and fiftieth anniversary of its foundation. In connection with these meetings, the importance and dignity of science were asserted in these eloquent words:

'Our children are born into a time in which science has already ceased to be a plaything; it has become, or is fast

becoming, the dominant factor in human affairs; it will determine who shall hold the supremacy among nations.'

So far, the note is one of exultation: yet the satisfaction of those who urge the claims of science is not complete. They complain that science is not yet studied as it should be; that other studies, whose value is inferior and whose day is over, are allowed to encroach on a field and share an authority which ought to belong to science alone. 'It has as yet,' the writer from whom I have just quoted goes on, 'no adequate place in the intellectual equipment or in the education of those who aspire to be the governing classes of the country.'

This sentence is significant in more than one way. Whether or not there are to be governing classes in the country—be that country England, of which the words were spoken, or America, to which they likewise apply—is exactly the problem which lies for solution before modern democracy. But however this may be, whether the nations will hand over their government to a trained class, or whether, according to the ideal impressed upon the United States by the founders of the Republic, the governing power shall comprise all classes and be the whole organised body of a self-governing nation, the claim is in either case made that science in its modern sense is to be the staple of their intellectual equipment.

Part of this claim has been already conceded. Immense endowments are lavished on scientific research and study. The axis of education has been sensibly shifted. Science has taken its place as an integral part of school and university education. The scientific methods of observation, record, and experiment have been introduced into other studies, and the scientific spirit, developed through the pursuit of the sciences, has become a general instrument of human culture.

Unhappily, however, this great and beneficent change has not taken place smoothly, or without grave conflicts and violent misunderstandings. Partly from exaggerated claims made by enthusiasts for the new learning, and still more largely from a narrow and obstinate conservatism among the supporters of the old, friction has ensued which is as needless as it is prejudicial. The idea has grown up that science is in some way opposed to art and letters. The unity of all knowledge, the co-ordination and mutual support of all human effort, has been lost sight of on both sides in this controversy. On one side were vested interests, old traditions, the jealousy with which innovations are apt to be regarded by those whose minds have been set in a particular pattern, and who cannot shift their perspective to the changes which the course of time brings about. On the other side were a revolt against the domination of these interests and traditions, a rejection of the stagnation involved in mere conservatism, and a necessary assertion of new needs and new methods of meeting them. But together with these came also an impatience of the past, an outlook narrowed by its own eagerness, and a recurrence of the belief that the path of progress lies in one, and in only one, direction. The fancied opposition of science to art and letters, and more particularly to poetry, is injurious to the general interest of mankind, to which all more special interests are subordinate. In a national life which executes its functions fully, science and poetry will not be in conflict, but in co-operation. Each corresponds to a need of life; in the full and harmonious development of life each reinforces the other; and in any sound system of national education both have their place, their proper and indispensable function.

We may regard this co-operation from either point of view: that which has primary regard to what poetry gains

from science, and that which looks, conversely, at what science gains from poetry. The creative instinct, the imaginative impulse, which find expression in poetry, are powerfully reinforced by the discoveries of science and by the growth of the scientific spirit. For that spirit affects the whole field of mental activity. The discoveries of science present the creative imagination with an ampler, richer, and more wonderful world; the spirit in which they are made and the methods by which they are pursued give a greater insight into that world. The scientific imagination is akin, though it works in a different field, to the poetic imagination. Both are creative energies; both work towards bringing out the organic laws of truth or of beauty which underlie the structure of man and of the universe in which man finds himself. The poetic imagination is, or ought to be, kindled by the work of science. The scientific imagination is, or ought to be, kindled by the work of poetry.

If we look to history, instances will at once occur where this conjunction has actually taken place. Ancient Greece invented science and perfected the art of poetry; and the development of Athenian poetry into what became, and still remains, the delight and wonder of the world, was coincident with the first growth, among the same race and in the same civilisation, of scientific enquiry, that is to say, of the search into the meaning and connection of things. The physical sciences were no doubt then still in their infancy: but the impulse towards them had been created and went side by side with the more patent and wide-spread impulse towards the scientific study of language and the operations of the human mind.

So too, at Rome, the great poem of Lucretius, in which Latin poetry for the first time reached its full stature, was inspired by the Epicurean philosophy; and that philosophy

was not only a system of ethics and a rule of life, but was—and was thereby distinguished from other philosophies—a systematic and brilliant attempt to solve the laws of nature and apply scientific principles to the construction and working of the physical universe. This scientific ardour was fixed by Lucretius as a poetic ideal. It was transmitted by him to his great successor in poetry. Virgil, in the celebrated passage where he gives utterance to his own ideal of life, prays that the Muses whose servant he is may before all else instruct him, not in the beauties of what is called a poet's world, river and woodland and a pastoral Arcadia, but in the 'causes of things,' the structure and law of the universe. Beyond poetry and beneath it lay the magnificent revelations of science; and only through the mastery of science could man enter into his inheritance, conquer fate, and dispel fear.

Once more, at the Renaissance, poetry and science found themselves working in close union. Each had a new birth; each gave the other mutual stimulus. Milton, in whom English poetry culminated, and who represents, for us as for his own time, the classic standard in poetical art, was a profound student of two sciences which in his age were making immense advances—those of music and astronomy. His scientific knowledge enriches and gives fibre to his whole poetry. In the 'Paradise Lost' he mentions only one of his contemporaries by name; and that one, it is significant to notice, is not a man of letters, but the most eminent man of science of that generation—the physicist and astronomer Galileo. Had he lived two hundred years later, we may guess that the name he would have chosen for this proud eminence would have been that of Darwin. Christ's College in the University of Cambridge, where both Milton and Darwin received their education, has lately been celebrating the memory of both. In that double celebration we may see

vividly not only the continuity and interconnection of learning, but the kinship of poetry with science, and the ideal of a university.

The expansion of science in more modern times has been concurrent with a similar expansion of poetry. The difficulty which both poetry and science have now to face lies just in this immense expansion of their field. Material accumulates faster than it can be dealt with. It is the day of the specialist both in science and in the art of letters. Against the narrowing effect of over-specialisation in his own particular field, the only safeguard is that width of outlook which is gained by grasping life as a whole, by mastering its pattern, as that pattern is discovered by the investigation of men of science, and is re-created or reinterpreted by the poets.

What poetry gains from science is strength and substance, a closer contact with the truth of things, and the power given by the use of a trained intellect. What science gains from poetry is something more impalpable, but not less important; it is what a French scientist calls *élan vital;* it is the impelling and organising force of ideas and imagination. Without ideas, pure science is little more than a record of facts. Without imagination, applied science is sterile. The earliest scientific theories were expressed in the imaginative forms of poetry: the latest are the application, to enormous masses of facts gathered through observation and experiment, of what may be almost called a creative insight, akin to, and based on, that imaginative power which is the essence of poetical creation, and which is fostered by the study of poetry. For by studying poetry we become partakers, to some extent and according to our powers, of the genius of the poets; we develop our own power of creative imagination. Now this creative imagination is not a separate fac-

ulty, shut off from the rest of our faculties. If it is treated as such, the results are disastrous: much of the suspicion and dislike with which poetry has been regarded among men of science is the natural result of a claim arrogated by men of letters, or by people brought up in the tradition of a time before science was recognised as a part of human culture and before scientific method had been applied to all the processes of life, that art and letters were the only sphere in which the imagination can work. But it remains true that it is normally through these that it is first kindled. It remains true that the study of science is most effectively pursued by those who approach it with an intelligence made sensitive, an imagination quickened, by the patterns of life created by poets and the pattern-making power which the study of poetry develops.

If there are defects in the present system of American education, they are due, according to the judgment of many thoughtful observers, to the fact that it hurries towards results without the wide preliminary training which develops the powers of the mind on all their sides. So far as this is the case, it condemns men to work with inferior tools, with an inadequate mental equipment. The result is like that of an engine racing: the mind is not in gear with the whole system of its surroundings, and much of its work is wasted. Energy and capacity are there in full measure; but the capacity has not the proper field to deploy itself in; the energy is forced to run in contracted channels, or, beyond these, to run to waste. Let me quote here the striking words used recently by a distinguished man of science and one of the most zealous advocates for giving science a primary place in national education.

'Several Americans have told me,' says Mr. A. E. Shipley, 'that comparatively few things are actually invented in

America, that most inventions come from abroad, but are eagerly taken up and exploited in the States. Where the American really shines is not as an inventor, but as a manufacturer. It is a striking fact that originality is rare in America, and I think it must be accounted for by the educational system. It stifles originality.'

This is a grave charge; but so far as the defect actually exists it should be realised, and so far as it is realised it can be remedied. We need to lay stress—and stress is being effectively laid by nearly all educationalists—on the necessity and value of scientific training for those who are destined to pursue art and literature. We need to lay stress likewise—and this need should not be neglected or postponed—on the necessity and value of literary and artistic training for those who are destined to pursue science. But to put it so is to state the case inadequately. For it is only a minority in an educated nation who will do either, whose life will be devoted wholly either to literary and artistic, or to scientific pursuits. Not only for these two limited classes, but for the whole of the nation of the future, the ideal which rises before us is that of an education developing all the faculties in harmony; of a nation brought into touch with the facts of Nature and her laws, and into touch no less with the best of what has been thought and felt by mankind and with its noblest and most beautiful expression. And this last is given us by poetry. Nature, as Bacon said, is conquered by obedience; and science teaches us the laws to be obeyed and the mastery over Nature which may be achieved by this obedience. Life is grasped and ordered by imaginative insight; and poetry teaches us the pattern of that order, and creates in us a new meaning, a new beauty and value, for the world and for ourselves.

V

POETRY AND BUSINESS

—

ONLY a few dedicate their life to the pursuit of science, only a few to the pursuit of art and letters. But we have all, in a greater or less degree, to do business. In it we have, directly or indirectly, our means of subsistence and our current occupation. Business is the substructure of life. A scientific community only means a community in which certain persons (comparatively a few) work systematically at science. They record their inventions or discoveries; they communicate the results of their research and the stimulus of their enthusiasm to others; and thus a certain secondary scientific knowledge, a certain appreciation of the scientific spirit and a large power of using scientific results, reaches through the mass of the people and colours the national life. A literary and artistic community only means a community in which certain persons (these also comparatively a few) do creative work in art and letters, and in the main body of which there is a certain appreciation of that work, and through it of the art and thought of other centuries and ages likewise. But a business community means one in which the whole mass and body of the nation, with insignificant exceptions, is engaged in business as its daily function, in which business is the staple of the national activity.

The United States are the greatest business community in the world. Industry and commerce have been, from the earliest days of the Republic, the chief pursuits of the nation, those to which it has applied itself constantly and

eagerly, upon which it has grown and thriven. On them the whole social fabric has been built up. With the vast increase of wealth due to expanding population and increased power of handling or creating material resources, the energy of business has kept increasing likewise, and its claims on life have become more and more imperious. A sort of fury of industry set in with the extension of the nation over the Middle and Western States, and just at the same time the great discoveries of applied science began to be made which have increased a hundredfold the control of man over nature. After the Civil War the reunited nation plunged into the business of material development on a scale and with a passion until then unknown in history. The business to be done multiplied faster than the hands who were there to do it. Everything became speeded up. Business encroached on all other national activities, and threatened to overwhelm the whole of life. Against this over-encroachment the national conscience is now beginning to rise up, and to reassert the claims of a smoother, less hurried, less perplexed life, not loaded down and breathless under the weight of its own machinery, but using that machinery towards ampler ends—as its master, not its slave.

Poetry and business may seem to have little to do with each other; or their relation, so far as any exists, to be one of mutual dislike and antagonism. Business methods are not the methods of art. The man of business is apt to regard poetry with contempt; and his contempt is fully reciprocated by many followers of poetry. Yet if both are necessary elements in civilised life, there must be some understanding to be come to between them, some harmony attainable. No poet can afford to neglect the machinery of industry; for by means of it he, like all other men, lives. But neither can the man of business afford (if he knew it)

to neglect poetry; for in it the life which he, like all other men, lives receives its meaning and interpretation. Business is a means, not an end. Its uses are necessary and great; but they require to be adjusted to ends beyond itself, beyond business for its own sake, if the life of the business man is to be one in which the full human capacities can be worthily employed. If his life is not touched and uplifted by imagination, he is the slave of business, and not its master.

For some, indeed,—and more perhaps in America than elsewhere,—business is more than an occupation: it is an art, and its exercise has a quality which might almost be called creative. The born man of business loves it for its own sake; and love implies some sort of ideal, some sort of exercise of the imaginative as well as of the practical faculties. Or we may rather say that the imaginative faculty, checked elsewhere, and not finding its natural outlet, forces itself into the one channel left open for it, and to some extent informs the life of business with ideals of its own, not to be scorned or denied, however short they may come of the higher and larger ideal. Without some such imaginative touch upon it,—and the touch is at best imperfect and rare,—how grey and joyless the purely business life is; how purposeless it seems in moments of serious reflection; how prosaic a world it offers! It keeps the world going, but at what a waste of the energies engaged on that laborious task! Let me quote what was said, sixty years ago, by an able man of business, a master of the theory and practice of finance. 'By dull care,' he wrote, 'by stupid industry, a certain social fabric somehow exists. People contrive to go out to their work, and to find work to employ them; body and soul are kept together. And this is what mankind have to shew for their six thousand years of toil and trouble!' These words of Bagehot are as true now as they were then. The human

race want more than to keep body and soul together: they want, and claim, not merely the continuance, but the fruition of life. Machinery to keep the world going is necessary; but it is not necessary, it is not right, that it should be kept going by turning masses of the nation into mere parts of the machine. For this would indeed be, in the noble line of a Latin poet, *propter vitam vivendi perdere causas,* 'for the sake of life to throw away all that makes life worth living.' It was not for this that man was created. It was not for this that the rights of man were asserted. To be enslaved to business is no less servitude than to be branded with the name and work at the caprice of a slave-owner. And as with the chattel slavery abolished by the Republic half a century ago, so with this subtler but equally real slavery to business (whether forced on the individual by circumstances or adopted by him of his own will under the illusion that it will bring him the real wealth of life), the evil effects spread far beyond the slaves themselves: they contract, degrade, and vitiate the whole life of a nation.

In common speech, as in popular thought, business is opposed to pleasure. This is highly significant. So far as the opposition represents a fact—and if it does not represent a fact, how are we to explain its prevalence, its being taken everywhere for granted?—it means that the unity of life has been lost. Business that does not bring pleasure with it, and in it, is only drudgery. It sustains life, but the life which it sustains is thin and barren. It accumulates wealth, but the value of wealth depends on the use made of it, and national, like private, riches are but the substructure of national well-being: they are the means of living, not the object of life. To bring business and pleasure into their true relation, business must be elevated from a mechanism into an art. This is not done by legislation: it is done by the self-realisation of

the human spirit. Towards this self-realisation poetry works; and therefore a nation needs poetry.

Business, or industry, has two sides—production and organisation. In order to elevate it into an art it must be carried on with pleasure and for the sake of pleasure. For this is the definition of art: it is production with pleasure and for the sake of producing. The pleasure of production is given by the pattern or ideal in the mind of the producer. And similarly, the pleasure of organisation is given by the pattern or ideal in the mind of the organiser. Now the function of poetry, as we have seen, is to create patterns or ideals of life; and the study of poetry means the reception into the mind of these patterns of life created by the poets, and their assimilation by the sympathetic instinct which they awaken. Thus received and assimilated, they fertilise life and make it fruitful; they make industry into a conscious pleasure. The beauty and the joy of life which they embody become part of our own life. Our industry becomes truly creative; our business is not carried on as a burden, but exercised as an art. Work and enjoyment are no longer contrary forces tearing our life asunder between them. Poetry, through the patterns of life created by the great poets, will raise us above our own lives, give us spiritual control over them, make the conduct of them no mere mechanical keeping of things going from day to day, but the daily exercise of faculties through which we are partakers in a full humanity.

Poets are often called dreamers, and some poets have been such. For the making of poetry is, like the other arts, also an industry; and, like other industries, it can be pursued mechanically: the poet may become absorbed in the workmanship of his art, and practise it, as the business man may practise his business, from mere habit, when he has lost the vital energy of creation. Or, like other ways of life, it can

be pursued with too much absorption; and, cutting itself away from the deep roots of thought, emotion, and experience, it may become a tissue of fantasies where the creative or imaginative powers have been working in a vacuum, and the patterns of life which they produce dissolve in the very act of forming themselves; as in some witch's weaving, 'the web, reeled off, curls and goes out like steam.' Nor is the study of poetry free from the same danger. Those who neglect business, which is the foundation of life, and conduct, which in the famous phrase is three-fourths of life, for the mere study of poetry as an art, may still find in that study both pleasure and occupation; but when thus cut off from what should be its foundation and substance, such study degenerates: it is apt to turn into the assiduity of the pedant or into the busy idleness of the dilettante. For those who content themselves with it—and all the more if by it they drug themselves into unconcern with activity and duty—the censure of the practical man of business is justified, and his contempt intelligible. They discredit the study of poetry by studying it wrongly. Not one of the least important functions which an institution of higher education fulfils is to direct and organise this study so as to make it really fertile, and to combine it with other studies in the scope of a training at once liberal and practical. The product of such institutions, so far as they succeed in doing what they set out to do, will be men and women nurtured among the ideals of thought and art, made sensitive to beauty, quickened by sympathetic intelligence, yet not so the less competent, but the more, to take their share in the business of the world, in commerce or finance or industry. A generation so equipped for life, and sent into it with the whole range of their faculties so developed, will not only keep the world going, but will raise the whole national life to a higher plane. They

will be in the highest sense good citizens: and in the goodness of its citizens lie the excellence and the true greatness of the state.

The ideals of citizenship include in them nearly all the lesser or more partial ideals aimed at through the specialisation of faculty on particular pursuits. By their wider scope and larger outlook they connect and balance these others. It is the privilege, as it is the duty, of a community which through the labour of past generations has conquered and cleared a dwelling-place for itself, to set in order and beautify its house. The pursuit of riches, of material comfort, even of greatness, is with the nation, as with the individual, a pursuit upon which the whole of life should not be spent. Until now the Republic has had her hands full with a great, necessary, and engrossing task—that of creating a nation, of organising a commonwealth, of bringing the resources of a continent under her control and asserting her place and dignity in the world. Upon that vast structure the spirit of beauty must be breathed, into it the patterns of noble thought, action, and emotion must be brought, to make the Republic of the future fulfil the plan of its founders, and justify the vast labour that past generations have lavished on building it up into material stability.

VI

POETRY AND DEMOCRACY

—

THE suspicion or dislike with which poetry is regarded by practical people, however unjust or exaggerated, has its reasons, and has existed in all ages and under all organisations of society. But in a democracy poetry lies under another special charge, which if made good against it would be fatal. It is regarded as the amusement of a leisured class, as something savouring of an aristocratic society. Art and letters as a whole share in this charge, but it falls on poetry with special force. Some kinds of literature have an obvious popular interest and make an obvious appeal to the mass of the nation. Some of the fine arts are applied directly, like architecture, to the public service, or directly affect, like music, the sensibility of massed audiences. Others are excused, rather than approved, because they employ labour, encourage special industries, and produce tangible material products. This is not the case with poetry. It stands or falls on its own merits, in its own inherent virtue.

But poetry is a function of life; and where life is organised under democratic standards poetry is, or should be, a function of the democratised nation. Much of the poetry of the past has been produced by and for a small cultured class. In aristocratic societies such a class was the pivot and guiding force of the nation; in it the imaginative ideals and the creative instincts of the whole people were concentrated, or, so far as they existed elsewhere, were used by it for its own purposes. The rest of the nation was but the soil out of

which that flower grew, or the fuel consumed to give the ruling class sustenance, ease, and material force ready to its hand. The public conscience now demands that there shall be no ruling class, but that all shall be fitted to rule. The aristocracy of intellect is subject to the same vices, and falls under the same condemnation, as the old aristocracy of birth, or the cruder modern aristocracy of riches. The ideal of democracy—far, indeed, yet from being realised, but felt everywhere, alike by its opponents and its followers, as a pressure steadily moving mankind in a particular direction—is that culture, like wealth and leisure, should be diffused through the whole nation. It abolishes the distinction between active and passive citizens, between a governing caste and a governed people. That is its political aspect. But its larger and nobler ideal is that of a community in which not only the task and responsibility of setting its own house in order and swaying its own destinies, but the whole conduct and development of its own culture, shall be universally shared; in which not only government, but life in its full compass, shall be conducted by the people for the people; in which the human race shall be joint inheritors of the fruits of the human spirit.

Only once, and among a single people, has this ideal been partially realised in the past. The democracy of Athens set no less an aim before itself, and for a brilliant moment seemed to have attained it. Poetry and art reached their climax there together with democratic government. It was the boast of Athens that culture no less than political power was shared by all her citizens. Poets and artists drew from that national atmosphere the creative and imaginative power which they embodied in their work, and returned to the nation in visible and immortal shapes the patterns of life with which the nation had inspired them. But the Athenian

democracy rested on insecure foundations. Like so many bright things, it came quickly to confusion, leaving behind it only a memory and an ideal to inspire all future ages. Many centuries had to elapse before the ideal of a civilised democracy was again raised as a standard before mankind by the founders of the American Republic.

The crimes and follies of the Middle Ages, it has been well said, were those of a complex bureaucracy in a half-civilised state. It is towards the end of the Middle Ages that we find the beginnings of national self-consciousness, and, with it, of democratic poetry, embodying patterns of national life. Nor was this all. As the inchoate or embryonic democracy began to be conscious of itself, it began also to be conscious of art, even when that art was the art produced among and for a limited class. As it began to be civilised, it began to have sympathy with the products of civilisation, and to take, if not yet to assert, some share in them. The ideal world of romance and chivalry opened out before it as something in which it could find patterns of life for itself. A common and universal religion, which in theory at least recognised no distinction between classes, between riches and poverty, between prince and people, gave a wide popular basis to all the arts which were employed in its service. Education began to leaven the community. Poetry sought and found a wider audience. Shakespeare produced his plays not for a literary class nor for a court circle, but for the populace of London who flocked to see and hear them. His own sympathies with the people have been doubted or denied; he seems, in the mouths of his characters, to speak of them with something like contempt. But he gave them a national drama. Even the epic, that stately form of poetry which has thriven in the courts of princes and deals with the high actions and passions of the

great, became in a wider sense national. The verses of Ariosto and Tasso, court poetry written for a highly-educated aristocratic circle, were sung by Venetian gondoliers and Lombard vine-dressers, as those of Pindar had been sung in ancient Greece by fishermen, and as those of Virgil are found scrawled on street walls in Pompeii. In England, Milton, a poet of profound learning and extraordinary technical skill, was read and appreciated not only by scholars or artists, but widely among a people whose study of the Bible had introduced them to literature and taught them in some measure to appreciate poetry. His genius penetrated and inspired the Puritan democracy; and though his own republicanism was of a severely aristocratic type, he may be called in some sense the source of republican poetry. For, once poetry had taken to do with the fate and destiny of mankind itself, it had to concern itself with the life and labour of the people as the main factor in human affairs. It found the reflection of the kingdom of God in the commonwealth of mankind. The freedom of God's ransomed drew with it as its consequence a freedom which was of this world. The equality of men before God bore with it their equality of rights and dignity here. The brotherhood of all God's children led on to the doctrine of a true fraternity, not only religious but political and social likewise, linking together all members of the human race.

The eighteenth century, that great germinal age of the human spirit, the age in which not only the American Commonwealth but the modern world was created, was one in which poetry held itself back. It was waiting for the shaping of the new structure of life: the task lay before it of fashioning that structure into new imaginative patterns, and giving it thereby organic form and vital interpretation. Towards the end of the century this preliminary work was well

on foot: the new world was taking substance, and lay ready for the transforming touch of the poets. The American Revolution had created the Republic. The French Revolution had shattered the old régime and its tradition in Europe. The Industrial Revolution was transforming the whole mechanism and texture of civilised life. In both continents a new world had begun. It was the world of the Rights of Man, of the *carrière ouverte,* of the sovereignty of the People; and into this world poetry let itself loose, to create, to interpret, to vivify. The idea of democracy had arisen among the thinkers and been translated into action by the statesmen; the patterns of a democratic world began to be wrought out by the poets.

Among the great English poets of that age, the greatest, in the combined mass and excellence of his work, is generally accounted to be Wordsworth. He divined the new age, but did not enter into it. His early democratic enthusiasm, chilled by the terrors of the French Revolution, became converted first into despair, and then into a search, in the recesses of his own mind, for ideals of life independent of external things. Yet he was the first, after Burns,—and Burns was then still only the poet of a small nation, not of the English-speaking race,—to link poetry with the requirements of nascent democracy. In his 'Lyrical Ballads,' as in the poems which succeeded them during his greatest period, he set himself expressly and deliberately to write poetry in the language of the people, and to seek the material out of which poetry was to be shaped in the common thoughts and passions and experiences of mankind.

Hardly less was the share borne in the democratisation of poetry by other great poets of that great period. Byron, himself an aristocrat by birth, believed in democracy; by his appeal to the elemental human passions he brought the im-

pact of poetry on the larger world which was prepared to receive it. Shelley reared before the eyes of that larger world the glittering fabric of an imaginatively reconstructed universe in which, freed from tyranny and superstition, from selfishness and apathy, the human race might develop its noblest qualities, and life be one long ecstasy of joy. Even those who regard Byron as a beautiful fiend, and Shelley as an ineffectual angel, must admit the truth of the striking words used of them by Tennyson, that these two poets, 'however mistaken they may be, did yet give the world another heart and new pulses.'

Even more striking and significant is the attitude towards an anticipated democracy, and the part to be played in it by poetry, which was taken by Keats. He was the youngest of that great group of revolutionary poets, the most gifted and the most splendid in his wonderful promise and unfinished achievement. Beyond all those others, with a width and foresight of vision all his own, he pointed and urged poetry forward. The horizon to which he saw is still distant and unreached. That 'joy in widest commonalty spread,' of which Wordsworth had profound glimpses, and which Shelley saw, as it were, through an iridescent burning mist, lay before the eyes of Keats, clearly, definitely, attainably. The world to which he looked forward was one in which, as he says, 'every human being might become great, and humanity, instead of being a wide heath of furze and briars, with here and there a remote oak or pine, would become a grand democracy of forest trees.' In that image he embodies for us the ideal of democracy in the highest and amplest form. And of this democratic ideal, poetry, because coextensive with human life, will be the informing spirit.

Democracy, we are often told, is on its trial. The brilliant promises of its youth have not been realised. It has

not transformed human nature. It has not done away with the vices of older civilisations, and it has developed new faults of its own. It is, among many of those who do not expressly reject it, accepted wearily as a necessity rather than embraced eagerly as a faith. Citizenship has with them become a burden, not an inspiration. Freedom and equality have sunk into mere formulary names, giving neither light nor heat, having little to do with the actual conduct and motives of life. Material progress goes on mechanically; the higher progress, the fuller self-realisation of mankind, is doubted or denied. Once more, as Wordsworth complained a century ago, false gods have been enthroned in the temple of the human spirit.

The wealthiest man among us is the best;
No grandeur now in nature or in book
Delights us: rapine, avarice, expense,—
This is idolatry, and these we adore:
Plain living and high thinking are no more.

So Wordsworth wrote then; and we must remember, if we are inclined to be despondent over the present case of democracy, that our dissatisfaction is no new thing, and that the mere fact of our being dissatisfied shews that we have not lost sight of higher ideals, and have the impulse in us, if we can direct and sustain it, to resume our progress towards them.

Poetry is also on its trial. The patterns of life it offers to us, the interpretation of life with which it presents us, seem to many unreal and remote. It speaks a strange language, thin and ghostly to the ears that are not attuned to it; it often holds itself aloof from, or mingles but passingly with,

the main current and texture of occupations and endeavours, of private pursuits or public interests.

Each alike suffers from the divorce that is between them. A democracy which excludes or ignores poetry cuts itself off from one of the main sources of vital strength and national greatness. A poetry which is out of sympathy with democracy is thereby out of touch with actual life. But the future that lies before both is splendid, if both will work in harmony, if national life is inspired and sustained by poetry, and poetry takes nothing less than that life for its province, gives it a heightened meaning, brings out from it the latent patterns of beauty after which it blindly but unceasingly aspires. Poetry, as Dryden said of it, is articulate music: the music to which life moves, and in which it finds its discords resolved.

Such is the task and function of the poets. But the study of poetry is not for poets alone, any more than the study of colour and form is confined to painters, or the study of music to composers. The appeal of art is universal. The inheritance of the present age is not merely the present, but the whole past as well. Of that inheritance, the great poetry of the world, from Homer downwards, is the most precious portion. It preserves for us, still alive and still having power to move and kindle, the best of what mankind has thought and felt, the most perfect forms into which it has cast its vision and reflection, its emotion and aspiration. And thus the study of poetry is part of democratic education; and the poetry of democracy, kindled by that study and appealing to a nation educated in it, will be the articulate music of national life.

JOHN WILLIAM MACKAIL.

THE SYSTEM OF THE SCIENCES

PRINCIPLES OF THE THEORY OF EDUCATION[1]

First Lecture

THE SYSTEM OF THE SCIENCES

AT the moment when the Rice Institute undertakes to begin its public work and to organize the wide and splendid educational activity for which it was intended, I do not know of any subject that could affect the development of its future more extensively and more deeply than the problem of the System of the Sciences. Judged by its present state, the problem appears more like a pastime for idle minds than a practical question of far-reaching importance. For at the present time there does not exist a single system of this kind, generally accepted and universally employed, and the numerous attempts of various investigators to establish such a system have not yet received the recognition which, on the one hand, would be a guarantee of its effectiveness, and, on the other hand, an indication that by the establishment of such a system one has discovered, in a measure at least, the correct and suitable thing. Nevertheless, such a system is a crying need, as is apparent at the present moment, when we are confronted by the problem of attaining a complete survey of all conceivable and possible sciences and similar activities of the human mind for the purpose of

[1] Two lectures prepared for the inauguration of the Rice Institute, by Privy Councilor Professor Wilhelm Ostwald, late Professor of Chemistry in the University of Leipsic, Nobel Laureate in Chemistry, 1909. Translated from the German by Professor Thomas Lindsey Blayney of the Rice Institute.

sketching a normal and rational plan for the realization of the fundamental aim of the Rice Institute. It may suffice at first, as has happened in the case of this institution, to select somewhat at random indubitable branches of science—*i.e.,* to let ourselves be guided by the demands and the attainments of our time—in order to be sure that at its organization at least a part of the entire range of science has been incorporated. But in measure as the present plans widen and it becomes necessary to envisage more accurately departments of this institution which are to be developed in the future, the need of a rational system that will include the functions of science as a whole will make itself felt more and more imperatively, and thus we shall be able only to postpone, but not to avoid, the question that confronts us here.

If we look about us to see how the problem has been solved by the universities which have been founded heretofore for the advancement of science, from an educational standpoint as well as for the purpose of scientific research and development, we shall find, generally speaking, the traditional four, sometimes five faculties. To the oldest faculty—the theological—law and medicine have been added, and all of the remaining sciences are united in the fourth—the philosophical faculty—which here and there, on account of its wealth of subject-matter, has already been divided into two parts, the one including the natural sciences and the other the so-called mental or historical sciences. If one raises the question as to the reason for this division, it will be seen that it is to be looked upon as a sort of fossil, as the fixation of a condition which belonged to the oldest period in the historical development of these institutions, and was in keeping with them, and which at the present time has completely lost its earlier significance. We know that all sciences in the early stages of their development formed

one great whole, which, together with all other departments of human activity having to do with mental work and cogitation, was intrusted to the oversight of a single corporation —the priesthood. A division of labor came only in the course of higher development, when the sum of all knowledge belonging to the single disciplines continued to increase to such an extent that it could no longer be contained in the head of a single person. First of all, the sciences relating to the regulation of human affairs and those having to do with the healing of human diseases were isolated, and they attained to self-administration. This produced the law and medical faculties. This process continued to be repeated under most varied forms, and we may see down to the present day how new sciences have detached themselves from the joint association in which they and other sciences have been included, and demonstrated their independence by providing their own chairs, texts, and curricula. This formation of new sciences has recently become so common and so varied that for a long while the universities have not seen their way clear toward providing each and all of them with opportunities for development. Hence there has come about a sort of division of labor between the different institutions in such a way that this or that special discipline is cultivated predominantly and with particular zeal in one institution, and other new disciplines in another. This does not depend in general upon systematic causes, but rather upon purely personal reasons. Whenever there is a gifted representative of a new discipline who is an excellent teacher and at the same time scientifically productive, he will be able sooner or later to acquire the means and influence to develop this new discipline into a recognized science. By surrounding himself with a circle of students suited to his purpose he sees to it that the local influence which he exercises in his home university

spreads within a few years over the entire civilized world, from which he draws his students and to which they return again after having imbibed the new ideas and new methods.

However gratifying the process may be in a given case, it is impossible to look upon it as the ideal solution of the problem in the general development of sciences. For by the side of the fortunate ones who at the place where they were accidentally situated succeeded in acquiring the necessary means for developing their new ideas and rendering them effective, there are many with whom things do not go so well. Those of us who are more intimately acquainted with institutions of higher learning, be it in Germany, England, France, or the United States, will recall those personalities who upon closer acquaintance revealed an astonishing store of new ideas and far-reaching plans, but who had not succeeded in gaining sympathy at the hands of the proper authorities for these ideas and plans, and who therefore were forced to exhaust themselves in unfruitful attempts to develop them and make their value felt. At all events, our institutions everywhere lack at present an arrangement for organization by means of which progress of this kind in all departments of science may be wisely encouraged and developed. As a result of this lack of organization science does not progress like a group of well-regulated workmen, cultivating a wilderness with new expedients and methods, putting it into proper condition for the proposed work of civilization, but advances rather like the bold individuals who moved toward the West in the early history of the United States, one settling here, another there, wherever accident or inclination led them, and where the character of the region or climate appealed to each one, leading a highly individual and peculiar life in the midst of manifold difficulties and dangers, paying little heed to what had been their connection with their homeland and

its civilization, for the extension of which they had undertaken their daring expeditions. In other words, we must give up the accidental development of science hitherto existing, which depended upon where and how, as occasion offered, the new disciplines came to light and found suitable soil for their growth, and substitute for it an entirely reasonable, systematic, and carefully considered type by means of which we may render the new soil productive for science. Thus we may organize our progress so regularly and systematically that it will march steadily forward and bring about a gradual improvement that is in keeping with the present condition of existing knowledge and adapted to the most pressing and immediate needs of future knowledge.

For this purpose it is absolutely necessary, in the first place, that we recognize exactly and clearly the legitimate relations which exist between the individual sciences, so that we may no longer be dependent upon accident for their advancement, but rather that by means of this conformability to certain laws we can indicate, and will have to do so with more or less precision, not only in which direction science must be extended, but also what, approximately, will be the character of the anticipated extensions.

If one were to inquire in what manner the problems arising here could be solved, the reply at hand would be that we must take from the previous historic development of science those determining facts which would serve as a criterion for the future development of science. In fact, we shall be convinced farther on that the fundamental ideas for the development of science at which we shall arrive are to be noted in the history of the formation of the various branches of knowledge. But an historical phenomenon is such a varied and complicated affair, however, that while those portions of it which conform to existing laws may be recognized when

the light of systematic knowledge is projected upon their differences, yet without such a guide recognition of what is authoritative by the mere observation of things heretofore existing appears rather hopeless. For, to begin with, one must of course convince himself that a number of accidental factors have encroached upon the development of science, particularly that all the sciences have sprung from the necessities of the hour and therefore have not been determined at their origin so much by systematic or general points of view as by the more or less urgent nature of the necessity and of the possibility of meeting it immediately or in the future—a possibility depending, to be sure, upon a variety of unforeseen circumstances.

If we examine in this sense the four traditional faculties, we recognize in the first three applied sciences. The first faculty—the theological—has to do with the content and form of religious tradition and of religious education, and is to this extent essentially an historical science which, however, cannot free itself to a certain extent from development in a modern sense. Thus law is an applied science, since its object is the regulation of legal relationships between people; and its functions are essentially historical, since the legal works of the later Roman Empire are still looked upon as the most important and in many ways the final source of law. The medical faculty represents, again, an applied science—viz., the technique of healing, and more recently the technique of avoiding human disease. Finally, in the philosophical faculty all is included that does not find a place in the three "higher" faculties, and in it pure or abstract sciences are found, such as mathematics and history, as well as applied sciences, such as dentistry and pharmacy. Thus we see that the historical and traditional division of the sciences as practised in the universities is totally devoid of system,

and that the original purpose of the universities to serve as training places for the future clergy still makes itself outwardly felt as a standard for classification and administration at a time when the theological aim has long since been relegated to the background.

The irregularities and inconsistencies, however, do not end with these matters; for, in addition to the three applied disciplines first mentioned, other large fields have been newly formed in the meantime—I mention only the technical fields for which a place has been provided in the universities only in a very incomplete and meager fashion. In Germany, therefore, the technical schools have developed independently of the universities, and have as a primal object the culture of modern technical applied sciences. By the side of these, quite recently, numerous other institutions have arisen, such as commercial academies, schools for administrative officials, and the like, which emphasizes the fact that universities, even with the inclusion of polytechnic schools, are at the present day no longer satisfying all the demands for the scientific treatment of important questions of life which our many-sided, prolific age has evolved. At the same time it is an expression of the fact that experience has proved the old system of science in the universities to be totally inadequate. Hence, from the purely practical reason that each nation must necessarily and primarily look to an organization of its educational system that will be as complete as possible and sufficient for the future, the need arises for utmost clearness in systematizing science.

From such an organization we may expect a better employment of resources existing heretofore, not only from the point of view that necessary disciplines which through accidental, external circumstances have not yet been developed will be taken into immediate systematic cultivation and

be made ready for their social functions, but also from the other point of view that certain fields which have been traditionally regarded as sciences and have been correspondingly supported by the government and have consumed the resources belonging to them will be shown by a systematic examination of the idea and meaning of science to be of very much less importance than has been admitted in the past. Thus we shall be able to free the present development of science from many narrow conceptions and trammels that have consumed the means, everywhere limited enough at best, for things whose social importance does not justify the employment of resources that have been raised for purposes of social betterment.

In a word, the problem is to replace the former disjointed and accidental development of sciences with an organized and systematized one. Like every other department of human activity, science also rose upon the basis of development purely individual. Those persons who felt a special inclination and special fitness for this kind of work endeavored to form their external circumstances so that they could carry on scientific work without coming too seriously into conflict with the requirements of life. And the general public, though it at first received the results of such disinterested work slowly and not without a certain amount of questioning, especially on the part of a church unfavorably disposed toward science, has more recently accepted them with increasing willingness and gratitude. We are now just beginning to emerge from this period of accidental scientific development. Numerous scientific institutions that were equipped hitherto chiefly for the purpose of instruction have begun very recently to develop exclusively with a view to the advancement of science, uninfluenced by any side interests, and thus in a most far-reaching way have assured the culti-

vation and dissemination of science in all civilized lands. Therefore in our day the question of national systematization of all science makes itself felt with special emphasis in order that its development may be organized—*i.e.*, may be subjected to thorough, wise, and judicious control.

For what is organization? What is the meaning of this process that has proved to be of fundamental importance in all departments of our present social life? The word relates to the existence of the characteristic desired in living beings, in organisms, and it is among them, in fact, that we find the principles in question put into practice and their existence long recognized. We know that a living creature is all the more perfect in proportion to its having been able to develop proper organs for the varied functions peculiar to its existence, and in proportion to its assuring more completely the common and organized co-operation of these organs by means of a central nervous system. In connection with all organization there come into question two related yet distinct operations: on the one hand, a division of functions and their apportionment to special organs for the purpose of having each single function all the more perfectly carried out by the particular organ formed for it, and secondly, a co-ordination of these single distributed functions in the interest of their common service in such a way that each single organ carries out its activities, in point of space as well as of time, so that it thereby produces the greatest gain for the whole organism. Therefore the distribution of functions and the combination of functions are the very essence of organization, and so we shall not be able to organize science otherwise than by separating its functions and then by reuniting them in collective efficiency.

A suitable division of functions implies, moreover, a knowledge of the separate functions—*i.e.*, it presupposes a

general survey of the total range of the sciences, and demands therefore a system of them, and this is shown to be the great practical problem that must be solved if we are to organize scientific progress logically.

One occasionally hears the objection raised that an organization of the sciences is not to be thought of, for the reason that science is the highest manifestation of spontaneous mental activity, and therefore is to be gratefully received, but should not be consciously and systematically directed toward definite problems and fields of work. Such an objection is not justified, for the reason that all human progress in all departments rests upon the fact that those things which have occurred heretofore unexpectedly and by chance are transformed into a systematized harvest in the field of human activity through our recognition of relationships established by law. Such an objection in the face of science has still less justification for the reason that science in its very essence rests, as we well know, upon the systematic, logical, and rational ordering of single facts. Therefore, only a very undeveloped condition of science as a whole is indicated if it has not yet learned to apply to itself this process of ordering of which it has always made use as a fundamental principle in connection with its own subjects of study. Thus we see that the ordering of facts and their relationships in each individual science is the first and most important function in its development. A discoverer of new facts may not content himself with simply imparting these facts to the world at large, but only after having recognized and fixed them does there then arise for him the new, great, essentially scientific duty of demonstrating the relationship borne by these new facts to the existing condition of knowledge in a particular field, and of thus rendering them real, organic component parts of the entire science in question.

An ordering process of this kind in each particular science has always been the principle of all progress, and the fact that great fields of possible knowledge have already been mastered by man and brought into natural relationship postulates the possibility (contemporaneously with the need felt for it) of our beginning to attempt the solution of the greatest problem of this kind. We are therefore confronted by the task of subjecting the whole range of science to the same organizing and systematizing process which has been carried out so successfully in single sciences, to the advantage of society as a whole.

Let us now, with the help of historic facts, endeavor to come to a clear understanding of the leading features of scientific development in order to arrive at certain fundamental ideas by means of which, independently of accidental happenings in the zigzag progress of historic development, we may extend the facts which have been discovered into an actual system of science. We recognize three chief phenomena in the history of science, which we shall discuss in their proper order, that we may derive from them those fundamental notions for our system which they enable us to reach. We have already called attention to the first of these facts, that all the sciences gradually separate from some central or general form of knowledge which at the dawn of history was everywhere in the hands of certain persons whom we usually term priests, but who are to be regarded more properly as the representatives of the entire knowledge of their time. Now that which above all else in the early course of historic development was concentrated in the hands of the priesthood was the guardianship of the supernatural relationship through which man imagines himself to be connected with the unseen powers, and the separation of it from those fields of knowledge which appear to him to be

natural relationships. These latter are those things which are subject to the law of causality, in connection with which one is accordingly able to recognize the conditions which must be fulfilled in order that the phenomena may be produced, and may be able to control more or less the course of these phenomena. This process of development is going on continually, and is far from being completely terminated in our own day. We may say, however, that on general principles we do not recognize within the entire range of science any field of supernatural phenomena; that, on the contrary, we are convinced that every phenomenon that can form in any way the content of human experience may be comprehended in its logical relationship to other phenomena, and may be co-ordinated thereby in the total fund of human knowledge. In contradistinction to the earliest and rudest conceptions connected with the belief in spirits and ghosts, this idea of the existence of a field beyond the reach of science has disappeared more and more rapidly from our mental life; and it may now be said, without prejudice to any person's individual attitude regarding the religious views of our day, that all fields of human experience are subject to scientific treatment; that therefore the idea of natural law is everywhere applicable; and that, under like circumstances, like consistency in resultant phenomena may always be expected.

The second point which an examination of historic development enables us to recognize is that the pure and abstract sciences grow by degrees out of the applied sciences. We have seen already that originally all mental activity was united in a totality of knowledge administered by the priesthood; or, more exactly, that in the hands of the priesthood only that kind of knowledge was formed which was not the common property of all adult citizens, and which was there-

fore used only on special occasions. On the other hand, the kind of knowledge necessary for the accomplishment of the tasks of daily life, of procuring booty, of combating enemies, of cultivating the soil and acquiring the necessary products for clothing, food, and shelter, could not be confined, of course, to the administration of a few persons, but was the common possession of all, and was apportioned from father to son with correspondingly slow increase, and from one member of the community to another. We know that just at this point a division of functions took place, just as we have seen in the case of the distinct kinds of knowledge which were administered by special classes of people. While in the original forms of social activity each person exercised all the functions necessary for life and maintenance, it subsequently came about, at first slowly but later in increasing measure, that the technique of division and combination of functions—in a word, the organization of management—caused certain functions demanding special skill—for example, blacksmithing—to separate and attain to their own degree of technical advancement, with their own traditions. With increasing cultural development, knowledge became here, too, increasingly diversified and richer, and the division of functions proceeded farther and farther.

All of these fields of knowledge, however, were only applied sciences, and quite an extensive special development inside the applied sciences was necessary before there were formed in certain advanced members of the human race new activities by which knowledge for its own sake, without any immediate reference to any application of it, came to be regarded as a vocation and aim in life.

European civilization began in this respect with an unusually rapid and brilliant development among the Greeks. There, owing to previous economic development and to the

formation of a small group of well-to-do men whose wealth was based on an extensive slavery, there arose men with sufficient leisure to direct their vision beyond the mere necessities of the day to more general problems and discussions. So we see that the early beginnings of pure science took the form of philosophic systems—in the first place, that of the Ionian natural philosophers. It was no longer a question of how one might satisfy the needs of the day more easily and to better purpose, but rather, since the needs of the hour no longer occupied these men, problems were sought in more distant fields.

Thanks to a well-known peculiarity of the human mind, the range of problems envisaged by them soon became extended to the utmost limits. Questions relating to the origin of the world, the manner and means by which living beings might have come into the world, and, after these, questions concerning the purpose and aim of human life, were the ones that busied these first thinkers. At the same time we notice that the pleasure of making use of this new organ of human activity—the capacity for reflection—soon led to extensive exaggerations in its use. Instead of supplementing in proper sequence the answer to questions relating to immediate surroundings, both as regards time and space, with solving more distant problems from remoter times and space, they ventured upon the remotest imaginable confines of time and space. It was only natural that these first activities of the newly developed thinking faculty in man should soon go astray in these distant and uncertain regions. A hard and long training was necessary before mankind learned that the newly grown wings could not, after all, bear them beyond the atmosphere of the earth, and that the first bold and illogical flights into unbounded space could only lead to the miscarriage of such impossible undertakings.

So we see how Greek philosophy turns back more and more from the excesses and capricious ideas of its early days to the realities of life and to an analysis of the capabilities of the human mind. We have received, unfortunately, only a very incomplete and highly one-sided and biased account of those days. Nevertheless it can be seen from these few literary remains that the Greeks had already entered upon a course which approached quite nearly to the modern development of the sciences, but which for that very reason was in complete contrast with the older traditions. This was the school of the Sophists, which demonstrated by its activity the inadequacy and complete inaccuracy of the first endeavors of youthful thought, and which, very logically, was accustomed to emphasize experience as the only reliable source of all human knowledge. These first germinations of scientific activity among the Greeks were in large measure repressed and destroyed by the great political upheavals which began some two thousand years ago.

Only a very small part of this mental stimulus was assimilated by the Romans and rendered fruitful of good; only a slender thread of tradition leads from those days, by way of Arabian translators and commentators, to the beginning of modern times, when the peoples of central Europe who had in the meantime become accessible to culture began to take part independently in the cultivation of science and of the reflective qualities of the human mind. The restriction of all medieval development to imbibing and discussing the traditions of the Greek philosophers resulted in the fact that during this time no new important intellectual productions were brought to light. In the same manner, the destruction of technical culture by the incursions incident to the migrations made it necessary that the stores of applied knowledge that belonged to varied fields of daily and social life, and

which were lost at this time, be slowly gained anew while a corresponding new technical culture on the part of these fresh peoples was slowly reformed. All these preliminary conditions were so far developed in the seventeenth century that at that time a phenomenon could take place similar to the one that had occurred among the Greeks a few centuries before the beginning of our era. For there arose again, on the basis of the general culture attained at that time, individuals whose thoughts were turned to science as such, and who, by systematically collecting what was known up to that time, put the human mind in possession of a disproportionately far-reaching power for overcoming terrestrial conditions. The historical appreciation of these events is rendered somewhat difficult, because at this time new sources of Greco-Roman tradition were opened, and especially because the artistic productions which had been found dating from the period brought before the eyes of modern artists new solutions of their problems that differed entirely from those which they had found previously in the course of their natural development.

This rehabilitation of Greco-Roman art in sculpture and architecture, as well as in poetry, is what one consistently terms renaissance. The learning, however, which developed in the province of mathematics and physics cannot be counted as a part of the renaissance movement. This development has in common with the former only the factor of time; it stood, however, in entirely conscious contradiction to rehabilitated tradition. While the artistic renaissance consisted in taking over the completed works of art of the past, as regards both content and form, and holding them up as unattainable ideals for the artistic movements of the day, the new sciences were in no wise developed as a rebirth of the sciences of antiquity, but rather in sharp and definite

contrast to them. It is characteristic that the Greek traditions in mathematics, for example, as contained in Euclid's geometry, did not lead to any kind of special development in geometric science; that, on the other hand, a new discipline that could not be traced traditionally in any way to the Greeks—algebra, and afterwards differential calculus—opened very extensive and important new fields to mathematics, and so, for the first time since those days, caused new scientific ideas and methods to appear in history as the original product of the peoples of central Europe. We find in the same way that the fundamental progress in physics, as brought about in the province of mechanics and astronomy by Galileo and Copernicus, arose in conscious and sharp contrast to the traditions of antiquity. In the truly fundamental investigations of Galileo concerning the mechanics of bodies falling freely, special reference was made to the false and untenable view previously held by force of tradition, and which was based upon the observations of Aristotle concerning this problem; and thus in a thousand other particulars can be shown the position of conscious contrast which the new sciences were forced to occupy toward the many traditions of Greek science.

In the few centuries which have passed since these beginnings a development in science has been accomplished which is incomparably greater and more varied than that attained in its first flower among the Greeks. And, judging from the progress which this highest attainment of the human mind has made, one may prophesy with certainty that the extraordinary development which has taken place down to the present time will be only the small beginning of incomparably greater further development, and that this science which, in the two or three centuries that it has been under the control of mankind, has already accomplished so very much in the

transformation and betterment of our life, will exhibit a much greater and more important range of activity still, both in the near and in the more distant future.

If we ask what this development means for us in respect to our chief problem—the system of the sciences—we cannot fail in general to recognize that an absolutely definite sequence can be shown in which the various scientific disciplines have appeared and have been developed into their first florescence. The first real science which we received from the Greeks was mathematics, especially geometry, and so we see also, on the occasion of the new flowering of science at the beginning of modern times, how mathematics stepped at once into the foreground of scientific interest. It reached at once such an unusual height of development with the discovery of differential calculus by Leibnitz and Newton, that all the performances of the past were left far behind, and our present mathematical knowledge still stands completely under its influence. This course of development is so characteristic that we can now say with certainty that the highest development of mathematical knowledge is already a matter of history. A wealth of unexpected, new results and prospects, such as the development of differential calculus and of its nearly related disciplines has brought with it, does not now exist in mathematics. Though year by year new progress may be noted in this oldest department of pure science, yet at the present time it is merely a question of extending and widening the fundamental ideas already existing, and we cannot mention a single mathematical discovery in the entire past nineteenth century that could have influenced the thought of the age in a way that even approached in importance and fruitfulness the discovery of differential calculus a century and a half before.

Following mathematics, astronomy—an applied science—

and physics were developed. This development began with mechanics, and was then extended to the fields of optics, heat, and electricity. We are not accustomed to recall the fact that the voltaic pile, for example, upon which the theory of the electric current is based, was not discovered until the year 1800, and that this entire field, therefore, which to-day, under the guise of electrotechnics, has so profoundly transformed our economic life, is scarcely more than a century old. Chemistry is even more recent than physics in its various disciplines, and began its scientific transformation only toward the end of the eighteenth century; in it, as in physics, we may experience from day to day the most astounding and unexpected extensions of our knowledge and views.

A whole series of other sciences—on the one hand, the biological; on the other hand, the so-called mental sciences, especially language, history, and finally sociology—was developed in the course of the nineteenth century and formed into sciences. This formative process is far from terminated, for sociology, as an example, is still occupied almost exclusively and above all things with inquiring upon what fundamental ideas its claims to being an independent science rest.

In this short review of the total development of science we can already see something like a system. We can say that mathematics and mechanics are about the simplest that we are able to discover in the whole variegated gathering of present knowledge, and that the sciences which appeared later, as of course was necessary psychologically, made their appearance later in proportion as their problems became increasingly more complicated. Unquestionably the problems of sociology, which has to do with the whole development of human culture, are disproportionately more complicated than, for example, the problems of chemistry,

which have to do with the reactions of objects without life and under uniform conditions.

We may derive, then, from these observations the three following facts. First, that we shall renounce in any scientific system the consideration of all supernatural relationships, of whatever nature, and that, on the other hand, from the very nature of things we shall extend our scientific problems to each and every field of human experience; secondly, that we must differentiate carefully between applied and pure (or free or theoretical) sciences, and in doing so we shall find the accidents of life and of origin principally in the sphere of applied sciences, whereas we shall look for theoretical and methodological relationships wholly in the pure sciences; and, thirdly and finally, that the general historical development of science also places at our disposal a clue for the systematic envisagement of all science by reason of the fact that from among the individual scientific disciplines the simplest arose and were developed first, and that, in proportion as the reliability of the human mind in mental operations was developed, the more complicated and diversified fields of experience were gradually submitted to science. We shall have occasion to subject to careful examination this last thought, especially regarding the increasing multiplicity of scientific subjects, since in this field we may expect first of all to find the solution of the problem regarding the rational systematization of all the sciences.

In order to find this general principle that has been sought, we shall first have to meet and answer the question, What is the general characteristic, the real essence, of science? It is evident that for the division of all science, only that peculiarity of it can be serviceable and effective which occurs to the same degree in each science, and which, therefore, is common to all. We may discover this common constituent

part all the more readily if we call to mind the origin of each individual science in such a way that we take into consideration not the special content of knowledge, but rather the manner and means of the formation of knowledge. Thus we now observe that every individual form of knowledge develops into a science after having been cultivated previously as a technique. The fact has already been emphasized that all sciences have had their origin in the needs and desires of life. From the fact that certain needs occurring frequently and regularly, such as those pertaining to food, the healing of the sick, administration, the building of homes, the making of clothing, etc., etc., have been met regularly from one generation to another, there have been accumulated a quantity of experiences which are handed down from father to son, from master to apprentice, and soon form a more or less important proportion of a particular science. This knowledge indicates in each case not only how things have previously come to pass in one way or another, but it points to what must be done in order to attain to any particular future results. Such a knowledge of the future is, for example, that in making bread one not only has to put the flour mixed with water in a hot oven, but that one must let the dough stand twenty-four hours or more beforehand, because otherwise the bread will not be sufficiently light. In the same way certain processes are worked out, for example, such as forging and hardening iron, and similar conditions may be found in every other field of knowledge. This means, in other words, that every branch of learning rests upon the knowledge of certain laws of nature, certain successions of phenomena, which are regularly repeated; and every technique is based upon the fact that one determines the hypotheses or preliminary conditions of every such succession of phenomena, so far as is desirable and suitable for

the work in hand, in order to bring the phenomena to a normal issue.

Every technical branch—and along with it, to an even greater degree, every science—has as its object, therefore, the attainment of future happenings by means of suitable preparations. It means, therefore, in the first place, foreseeing the future, hence forming the future. Both possibilities have their limits. One can foresee the future only in part and for a relatively short duration of time, and one cannot prevent many approaching events, even when it may be desirable to do so, because the means for altering future occurrences are more circumscribed than those for foreseeing them. But, nevertheless, the number of things which may be foreseen and influenced is continually increasing in proportion as knowledge, and therefore in proportion as science, reaches farther.

All prognostications of this kind depend for their part upon the circumstance that certain groups of phenomena always occur conjointly. The groups may not be connected as regards time; in such cases it is a question of objects, or subjects, of our experience such as are included in the nouns "horse," "stone," "fire," "sky," etc. Each of these words indicates a definite accumulation of experiences, repeatedly gone through with, which have the special characteristic that as regards time they are always to be observed connectedly. These peculiarities that are observed coincidently are called the characteristics of that particular thing, and the technical as well as scientific knowledge of this thing is all the better and more developed in proportion as characteristics and relationships are better known. The most frequent and best known of these kinds of groups are given definite names, as we have just seen. Those which are less well known and which have been investigated only in the

course of conscious endeavor are expressed more commonly through rules and natural laws. However, in both cases it is a question of the relationships of definite single happenings or single characteristics; and prognostication depends in any case upon the fact that, after taking cognizance of some few of these characteristics, one finds himself in a position to predict the others also. Thus the primitive huntsman, for example, contents himself with the optical picture of game well known to him in order to set out at once to pursue or kill it, though he has not been able to discover from experience that it can be killed and transformed into food. Since, however, this experience has been met with in connection with similar things on previous occasions, he dares prophesy that it will also be so in this case; and this prognostication, therefore, is sufficiently certain for him to expend trouble and labor upon the killing of the game. In a somewhat more advanced period of cultural development this ability to foresee events is even extended considerably further, in that man intrusts seeds to the earth with the foreknowledge that in proper time they will grow, that the resulting plants will bear fruit of like structure and in such quantity that the measure of grains used in sowing will be abundantly replaced.

In this manner one can work at will through the whole range of human activities and knowledge, and the general fact will always be encountered that all conscious performance rests upon a knowledge of the regular temporal relationship between different experiences which recur in the same way. Thus all knowledge consists in group-memories by means of which certain definite amounts of experiences, happening simultaneously or in sequence, are included from time to time. Through the general psychophysical nature of all living beings, repeated experiences affect the individual

experiencing them otherwise than do single or varying ones. They become possessed of a special characteristic which, in connection with conscious living beings, we term remembrance or acquaintance. Upon this recollection of regularly recurring associations, or our acquaintance with them, depends our power, at first more instinctive, later more conscious, of foreseeing and anticipating future events. Such associations we may comprehend under the general term of concepts, in connection with which it is well to repeat emphatically that natural laws are also to be classed under concepts; for they represent relationships, just as the ideas "horse" and "stone" represent associations of definite occurrences that may be experienced, or characteristics connected with the object in question.

Such a formation of ideas has a purely technical character at the beginning—*i.e.,* only concepts impress themselves in the consciousness of primitive man by repetition and a corresponding awakening of interest as relates to experiences which are important to him for his existence. He could form for himself, for example, ideas or experiences concerning the many thousand plants which he has the opportunity of observing daily. He confines himself, however, to those plants from which he derives a special advantage or harm, and avoids the forming of ideas concerning less urgent objects, for the reason that they have no known importance to him, and because for that reason he shuns the efforts (and evidently they must have been very great in the case of primitive man) necessary for such a formation of ideas. One may observe this condition of mind in all possible gradations among races which are but slightly developed, whose characteristics and psyche have become well known to us in recent time, thanks to the many anthropological investigations. We may thus observe all the stages, from the most

circumscribed formation of ideas, which extend only to the most urgent necessities of life, up to the very highest development in this sphere, such as may be found in the mind of the modern investigator, discoverer, or organizer.

These considerations now lead us to establish the essential difference between technique and science, or, as one may more properly express it, between applied and pure science. Technical knowledge, from its very origin, gravitates around certain necessary or desired things, and all knowledge is created and collected in respect to the accomplishment of the task that underlies this relationship. Whenever a technique is followed, however, for a considerable length of time, and is developed to even greater completeness, it always happens that new circumstances arise from time to time which cannot be controlled by existing knowledge, but which demand rather the acquisition of new knowledge. The more varied the knowledge as regards the events which occur more infrequently is, the more experienced, the better informed, the wiser the person in question is. The necessity of being prepared for unforeseen cases causes finally a certain inclination of mind in accordance with which, even without any thought of a particular task's lying just before one, a condition of preparedness for all possible problems appears to be a desirable state. Accordingly, one will strive not only to become acquainted with the material employed by the technique in question with reference to its immediate application, but one will endeavor to investigate it from so many sides and so variously that the future occurrences in that technique may, in so far as is possible, offer no further surprises of any kind. It should not be stated that this train of thoughts is the only one which has led from technique to science. But, so far as we can historically see from the statements of those who have created a science out of technique,

this general impulse to know more than the needs of the hour require, and to be prepared for all eventualities, has been, after all, a prime motive power everywhere for the carrying out of research work.

Since the necessities of life are always transformed, according to the well-known laws of natural selection, into activities which promote happiness, because the beings that gladly and readily perform the necessary thing are, in the struggle for existence, especially preferred as compared with others, it is also to be expected that the necessity for logically controlling phenomena becomes by degrees a passion for knowledge. This passion for knowledge is a variation of the racial type, the origin of which is therefore to be expected only in a very few extraordinary individuals. And so history teaches us that the investigator, the man who, independently of any technical application, though possibly incited by it, feels the general impulse to extend his knowledge and to shape it into greater effectiveness by a process of rational comprehension, was originally a sporadic phenomenon. It is to so small a degree a question of professional investigation, that in the case of the first investigators in each special department we cannot help observing just about the opposite. Those who content themselves with handing down existing knowledge look upon every extension and renovation of intellectual materials as a wrong done their efforts, and take a most energetic stand in opposition to any possible change in pre-existing functions. So these greatest and most decisive benefactors of mankind, the men who have endeavored to transform the short-sighted technique of their day into a correspondingly more far-sighted science, have almost always been persecuted and oppressed. And though investigative activity in our day is no longer fraught with danger to life as it was three or four centuries ago, and

though in our scientific institutions attention is generously and readily given to carrying out investigations, we see, nevertheless, that even in the twentieth century the profession of the investigator as such is found only sporadically as yet. Professional appointments of this type exist in the form of research professors in the American universities and members of the research institutes of Germany, and certain professoriates in the old English universities, Cambridge and Oxford, as well as a number of professoriates in the Sorbonne in Paris. All the other positions in which research work is now being carried on permit this work only as a species of minor office, the men in question being appointed either as professors for the instruction of students or for some other regular activity which, to be sure, has some factitive relationship to their research work, but is quite secondary to it so far as their outward position is concerned. To find means new and universally applicable by which research work may be regularly overseen and encouraged by society, whether by the government or by narrower groups within the government, is a great task for the twentieth century and for this institution now in process of formation; the question of a logical separation of instruction and research will also be of vital import, if the Institute is to attain to the high aim which its founders have set for it. In other words, the relationship between research and teaching must be organized. Each of the functions must be developed to the highest possible point of efficiency. And since along with a division of functions the co-ordination of functions is also important to all true organization, care must be taken that a member of the Institute who is occupied chiefly with research work shall have an influence upon its entire intellectual activities which is proportionate to the extent of his ability and his tasks.

We have seen how the Greeks, immediately after the discovery of the enormous power that comes with the development of ideas and laws, in the freshness of their youth soon greatly exaggerated their view of the effectiveness and productiveness of this new intellectual instrument. Instead of allowing the formation of concepts to depend exclusively upon experience, as the very nature of the matter demands, the Greeks, as soon as they had experienced the workings of abstract intellectual activity, attempted to increase it to the very furthermost limits. Instead of forming for themselves conceptions about the nature of the earth, about the laws governing the growth of plants and the propagation of animals, about weather and clouds, in keeping with actual conditions, they soon extended their speculations to the most universal and unattainable problems, such as the beginning and end of existence, the nature of the entire visible and invisible world, and thus took as a subject for their meditations those ultimate characteristics of bodies physically perceptible which are far beyond the confines of perceptibility. Such intoxication in the use of this newly discovered intellectual power is readily explicable, but we must always keep before our minds the fact that it is only a species of intoxication and exaggeration of a means newly won with which we have to do, and we must not consider the intellectual accomplishments in the sphere of speculation that date from that youthful period in man's development as unassailable and unsurpassable master-accomplishments of ripe intellectuality.

Greek speculation is an expression of childlike pleasure at the new intellectual acquisitions. Just as a child, after having overcome the first difficulties of speech formation, cannot repeat and vary the art just learned enough, so, too, in connection with the Greeks we see the theoretical or abstract

thinking—*i.e.*, that which has only its very earliest origin in common with the necessities of life—developing in a great variety of forms, but leading only in rare cases to a lasting and really fundamental result. So in geometry Greek thought created a theoretical or pure science (probably in close conjunction with the empiricism of the Egyptian surveyors), which found its classic expression in Euclid and in this form influenced most profoundly the later development of the science. Since the Euclidean form, however, is only the product of a very long and thorough study of this science, and, therefore, does not consciously contain the slightest trace of its genesis, this presentation of geometry is anything but suited to introducing the formative mind to the way in which science has its origin. It is only typical of the manner in which science, which has already developed prosperously to a considerably advanced stage of completeness, may be logically co-ordinated in accordance with known principles.

An insight into this relationship is essential to all the questions pertaining to instruction and education. The conception which has been current for centuries, that the geometry of Euclid is an especially good means of training for the human mind, is incorrect in so far as one places any important degree of weight at all upon the development of the capacity for discovering new relationships and principles, for forming new concepts out of the chaos of varied experiences for the advantage of mankind. For this there is not the slightest introduction in Euclid.

So we see, then, that after the overthrow of medieval barbarism, through which there ran only a slender thread of earlier cultural traditions, the entire science of the time was occupied with the task of developing this tradition as completely as possible. The real key to it was the knowledge of

ancient languages. For a long time men contented themselves with Latin writings, until finally, owing to a series of accidents at the beginning of the sixteenth century, the Greek language and Greek tradition became known in central Europe and gave impetus to great movements which were called the Renaissance of art and letters. This Renaissance had to do primarily with art; secondarily, under the name of humanism, with literature, which, by way of the long circuitous route through Arabian translations, became known to the peoples of central Europe, who were once more struggling upward. Independently of these, modern mathematics, then physics and chemistry, arose, as we have already seen from what has been said.

The fact that, owing to the Renaissance, the predominant occupation of the time was on the language side of classic tradition was brought about because the ancient works in the course of their transmission had undergone extensive disfigurement and harmful changes, so that the reconstruction of the original, genuine text was an important preliminary condition for attaining their real content. As is almost unavoidable in such cases, the means was gradually made the end, and the treatment of the corrupt and disfigured texts by means of the apparatus of linguistic criticism became, without any consideration for the content that might possibly be arrived at thereby, the subject-matter of zealous and devoted labors, whereby the results stood completely out of proportion to the efforts expended. Through this circumstance a thread of pseudo-science developed by the side of that thread of real or empirical science which has been described above.

In our universities there may yet be found a great number of men occupied with the same tasks with which culture, just beginning to emerge out of barbarism, was compelled to busy itself at the beginning of modern times, in order to

disclose the only fountains of culture existing at that time—the traditions of classical writings. At the present time, in all sciences without exception, we have far surpassed the stage reached by the Greeks and Romans. An objective, therefore, for expurgating the traditions from those days, by means of the apparatus of philological criticism, no longer exists; owing, however, to the law of the conservation of form, this work is still being continued down to the present day. And a goodly portion of the respect that was paid to this activity, and with some degree of justice, three or four hundred years ago has been maintained down to our time, when work of this nature has completely lost its former importance and has not in the meantime attained to any new significance. I know full well that with such notions I am placing myself in contradiction to the majority of my contemporaries who are studying the problem of the sciences. But at this important opportunity I cannot refrain from calling attention, with all possible emphasis, to the conclusions at which I have arrived on this question. The whole province of the so-called mental sciences,—above all, classical philology, and in connection therewith many other historical disciplines, are nothing but passing phenomena which do not proceed from the continual ascent in the development of civilization, but rather from transitory waves in this great stream. In their own place, in so far as it is a question of the proof of certain stages of culture and their peculiarities, this knowledge has still a certain value; but it stands upon the same plane of importance, for example, as the knowledge of the development of the ancient Mexican civilization, or that of China or any other, and it is out of the question to attribute indefinitely this predominant importance to the history of the development and content of Greek civilization which by force of tradition we still concede to it.

And in measure as the general laws of the development of culture become better known (we shall return later in proper connection to this problem), in the same proportion also the knowledge of a particular case will lose in importance. For when the general law is known all the individual cases are known along with it, and there is no need of studying a special case more thoroughly than is absolutely necessary for the clear and definite ends in view.

To whomsoever this judgment may appear severe or unjust, I would beg him to call to mind the fundamental definition of science at which we arrived in our study of its historical development. Science exists for the purpose of prophecy; a science, however, which confines itself to gathering information in the most exact possible way concerning the minutiæ of some past epoch in the development of a certain people—as, for example, Greek archæology does—foregoes from the very start all claim to the character of a science and confines itself to representing a branch of knowledge that possibly at some later time (when it becomes impregnated with more general interests in the wider field of mental activities) may serve as material for a science, but in itself is in no wise a science in the general modern sense. The same thing is true of the relatively modern discipline of comparative philology, which also has limited itself in its functions up to the present exclusively to determining what has existed, or at least still exists, with respect to the various exceedingly diverse and therefore accidental forms for the signs—but slightly subject to intelligible laws—which different groups of humanity have co-ordinated with the ideas they have formed. However well one may know the past and present of these formations, one has not attained in the slightest degree to anything in the nature of true science. Such a thing could occur only if one employed the

knowledge of the past and present for the prevision of the future, and, whenever possible, for preshaping it. This leads us to the true task of linguistics, namely, to the task of setting in place of (or at first by the side of) the more and more impossible multiplicity of national languages, which have originated within narrow circles, a new general language, free from all the imperfections and shortcomings possessed without exception by all those formations that have risen by accident, and uniting all the advantages and special auxiliary means that it is possible to observe in the individual languages for the fulfilment of the purpose of attaining an exchange of thoughts as rational, simple, and unequivocal as is possible. In the same way, chemistry was an exceedingly incomplete science,—indeed, it was only the beginning of one so long as it confined itself to the analysis of the materials at hand and to the determination of their characteristics. Chemistry became an all-transforming science only after it had learned, on the basis of the knowledge so obtained, to form limitless quantities of new materials with new characteristics,—in fact, to seek consciously and to gain synthetically materials with definite characteristics, concerning the existence of which nothing was known up to that time, but whose manner of production and whose presumable characteristics could be foreseen on the basis of the scientific knowledge already attained. In precisely the same way we must employ our present knowledge concerning the formation and transformation of language in order to construct a really complete and universally available language which may serve for the general intercourse of mankind, at first by the side of, and perhaps at a very remote date exclusively in the place of, the national languages which have arisen by accident.

These considerations, which in an analogous way we may

extend to history (whose present scientific representatives for the greater part also still refuse to employ their knowledge of what has transpired in the past for a logical predetermination of the future), teach us that the group of so-called mental sciences at present correspond much less to the real substance of science than do the natural sciences. Nevertheless, we observe how the natural sciences, ascending from the simpler to the more complex, become more and more imbued with scientific method and with the true idea of science, which looks toward prediction, and the approach nearer and nearer to the ideal thus characterized. So physics and chemistry in large measure have already reached this stage, while biology is most zealously engaged in endeavoring to attain to it. And in very recent times we have seen a group of sciences adopting the same methods, and with their assistance making great strides in the direction of human progress and culture. I refer to the cultural sciences comprised at present under the name of sociology. The mental sciences, of which we have just spoken as being undeveloped, are gradually being taken over by sociology, and are being fructified and rejuvenated by the application of general scientific methods. And so in our own day a renaissance of science is beginning to be felt, which, however, unlike that of the artistic Renaissance, does not confine itself to a relatively short space of time, but rather began three hundred years ago, after the rebirth of the sciences in central Europe; and it has experienced, and must still experience, especially at present, great and important changes in the entire thought of the time (as example I mention only the transformation of our conception of the nature of legal relationship due to the irresistible socialization of jurisprudence in our day).

These observations lead us now with absolute certainty to the chief point of our problem,—to the question, According

to what principles are all the sciences to be divided? We have seen that the element common to all the sciences is the formation of ideas and the investigation of the relationships between the ideas thus formed. And we shall have to seek, therefore, a basis of division for all the sciences in the nature of the ideas with which the various sciences are busied. We see at once that such a division cannot be employed in connection with the applied sciences, which are a product of the physiological and psychic requirements of the human race and of the accidental climatic and local conditions incident to its development, but that a system of this kind can be found only in connection with the pure sciences, which are independent of such sources and motives, and which are directed merely toward the solution of relationships belonging in the field of concepts. Just here an exceedingly simple and perspicuous system presents itself, which provides us with the frame within which all human knowledge may be logically and methodically included, both in the form in which it exists at present and in whatever form it may assume in time to come.

When we consider, therefore, the various ideas which humanity has formed, and which have been brought into order by science, we find in them the following fundamental difference. There are certain ideas which are the fairly immediate results of experience, and which have retained in consequence a relatively large proportion of the inexhaustible diversity which every experience brings with it. Such concepts, for example, are "man," or "tree," or "government," etc. Since, however, the entire content of a single experience does not serve each time for the formation of ideas, but only those common portions of each experience that occur in a great number of them, therefore every idea is poorer than the single experience which can be associated

with the idea. In every single horse one is able to show more individual differences than are contained in the general idea "horse." Hence we must take no note of a certain part of every experience, or, technically speaking, we must abstract from it, in order to arrive at the general idea in question. Now this deduction may be carried more or less far,—it may be carried so far, for example, that the idea "horse" is still retained; it may, however, be carried, if we include "horse," "dog," and "butterfly," as far as the much wider term "animal," whereby we abstract from the special peculiarities of particular animals, and take account only of certain common peculiarities, such as assimilation, oxidation, locomotion, reproduction, etc. One can imagine this process of abstraction extended until we shall finally arrive at notions which are applicable to practically all experiences, but which on that account have sacrificed most of the peculiarities of each single experience. In fact, they can retain only such peculiarities as occur in all experiences, and which, therefore, are of the most general character possible.

Thus there will always exist a reciprocal relationship between the diversity of the single characteristics, or parts, included within an idea, and the number of experiences or things in general which can be brought within this idea. The richer an idea is as regards content, the smaller it is as regards its range, as regards the number of individuals that come under this idea, and *vice versa*. This is a relationship which is universal and which therefore represents the principle sought for in the division of ideas and thus in that of the sciences. We begin with a science having reference to ideas of the widest range and least content, which accordingly predicate something about each and every experience, but can make only circumscribed and very general predications about these experiences. We can then ascend to con-

cepts which have a richer content, but which on that account also refer to a narrower range of experience, and we can thereafter continue this process step by step. Since every idea must have a definite wealth of content and a definite range, which must fall between the furthermost limits of the most general and far-reaching ideas, on the one hand, and of the richest and narrowest concepts on the other, so we see that according to this principle we can actually dispose of all ideas, each in its own place, and that a systematic arrangement of all conceivable and possible sciences, in the order of narrowing range and increasing content of the ideas, gives us the certainty of logically encompassing all human thought and hence all the human sciences possible.

Before we undertake to carry out this general idea, there are perhaps a few words to be said about the real task of science in connection with the study of concepts. In accordance with what has been said, in order to have the totality of science, it should suffice to enregister all existing ideas in accordance with this principle of content and range, and to group similar ideas. Such a notion is, of course, entirely incorrect, and the error arises from the fact that we have tacitly considered the materials of our ideas as being complete and correct. Actually, however, no single idea represents an enduring and unchangeable image; rather is it constantly subject to new treatment, owing to the development of special knowledge and to the increase of our experience. This development, on the one hand, takes the direction of causing us to discover new elements of the idea which previously had not been known. For example, every investigation of the action of any substance in chemistry produces new facts of this kind, which contribute to a more exact characterization of the idea of the particular substance—*i.e.,* to a more extensive differentiation of its content. More-

over, the ideas as first formed by man have not in many cases been grouped and delimited most adequately; and there is a second kind of concept-making going on continuously in all science, which consists in our so altering the range and content of the idea as it was originally determined, and so analyzing it or grouping it with others, that a more logical —*i.e.,* a more distinct—division and arrangement of the ideas are rendered possible in a way calculated to bring out more clearly the existing relationship. It is thus that we gradually approach the solution of the standing problem of science, namely, by rendering as innocuous as possible the effects which the process of abstraction, in necessarily limiting experience, produces in every concept; by endeavoring to emphasize as completely and as diversely as possible the elements of the idea which remain after this process of abstraction, and to determine from every point of view their present relationship within the range of the idea.

Furthermore, there exist relationships between the various ideas which had not been recognized at the time the concept was formed. To discover these relationships is yet another exceedingly complicated and varied problem of science. It forms the so-called deductive part of it, while the determination of the ideas and their content is usually called the inductive process of science. Our results are not very satisfactory when we attempt to represent these two kinds of activities as opposed to and independent of each other, for real scientific work results from the uninterrupted employment of both methods. But few sciences have developed to such an extent that the deductive part has gained ground, as is the case, for example, with geometry and already to a certain extent with thermodynamics.

So we have now made the necessary preparations in order to undertake in detail the formulation of the pure or abstract

sciences. For this purpose we shall next endeavor to find the very widest general idea with which any experience or object one pleases may be co-ordinated, which therefore possesses the greatest compass of all conceivable ideas, and in addition, of course, the least imaginable content. This concept has no definite name, for its establishment is necessary only for the purposes of pure science. In every-day life a concept so comprehensive and so poor in content finds no suitable application. We shall therefore experience a certain amount of difficulty in designating adequately this concept with the help of language. Whether we speak of a thing or an experience, of an object, or of anything else which approaches this concept, we run the risk of ascribing too great profuseness to its content, hence too much narrowness to its compass. We shall, therefore, content ourselves with the description that this most general idea—to which, in order to be able to speak of it, we shall ascribe the name "thing"—has no other characteristic than that it represents an experience which can be differentiated from others. So long, indeed, as all experience is felt to be a regular, invariable sequence of situation, there can evidently be no question of any conceptual activity. Only when the different portions of our experience react differently upon us, and those which are similar and coincident are included to the exclusion of the others, do the first traces of conceptive activity appear. Thus there occurs automatically and unconsciously the differentiation of our experiences and the arrangement of the corresponding parts, owing to that general characteristic of living beings which has been called by Hering, in the widest sense, memory—the basis for all concept-building in general. In this sense a thing is everything of which we are aware and which we can feel to be different

from other things—a thought just as well as a house, a sensation of pain as well as the Milky Way, etc., etc.

A science of this thing alone, without further content, is impossible; for all that can be predicated about it is limited by our definition to the fact that we can differentiate it from others and recognize it again on the occasion of its reappearance in our experience. In order that a science be possible, we must therefore be able to bring somewhat more content into the idea. This content consists, in the first place, in our not limiting consciousness to a single thing, but in combining a number of things which appear as belonging in any way together into an association or group. As soon as we do that we get at once a whole number of possibilities of testing certain experiences in connection with such groups, and of setting up certain laws of nature which express these experiences. If, for example, we have, on the one hand, a group of children, and, on the other hand, a number of apples, we may give each child an apple and we are quite certain that one of the following cases will arise. Either each child receives an apple, or, after giving out the last apple, some children are left over, or there are apples left over after the last child has received an apple. There can be no other situation—*i.e.,* in other words, we have never experienced the occurrence of other possibilities when we co-ordinate the members of one group singly with the members of another group, as has just been described.

The usual way of expressing this law—that, of two things, one is either like or greater or smaller than the other—is somewhat too narrow, because the idea of number is included, which we arrive at only in connection with a later development of the considerations we have suggested here. In the same way, the philosophical law regarding the ex-

cluded middle, which is recognized as one of the fundamentals of logic, is only a special case of the general law of co-ordination, whose relation to the general law I need not explain further. At all events, we see that even in the case of an exceedingly general operation, as in the case of inclusion of things into groups and of the co-ordination of these groups with one another, very definite peculiarities conformable to law soon appear, which we find again in every single case, whatever may be the content of the group in other respects. They have, therefore, no distinguishing mark other than that one can merely distinguish one from the others. We may sum this all up in the statement that, of two groups, the one must either be similar to or richer or poorer than the other; other cases than these three never occur, and are therefore, as far as experience goes, impossible. These experiences are so very frequent and so common that we cannot imagine a world in which these simple laws of co-ordination do not hold. Accordingly we have not considered these laws as empirical laws, which they really are, but as *a priori* laws which inhere in the human mind before any experience, but which do not come into its consciousness until after it has encountered them through individual experience. We have no reason to hold longer this artificial construction, which has no real basis, but whose source is to be traced back to the half-forgotten religious conceptions regarding the act of creation and the endowment of man with certain characteristics on the occasion of that act.

With these considerations others may be connected by means of which one arrives at the idea of number by the comparison of groups and by the systematic construction of them out of single numbers. We see, then, that if we confine our observations to enumerable things, we arrive

at a corresponding science—arithmetic, or science of number. Some one may perhaps ask whether there really are any things at all which cannot be counted. Without doubt, an affirmative answer must be given to this question; for in our experience we have a number of parts which are variable without our being able to set the various single parts over and against each other. When, for example, we look at the sky at sunset, it has on the horizon the color of gold and gleams; as one looks toward the zenith, this gleaming phase of the sky passes by degrees through greenish tones over into the pure blue of the sky. In this case we are quite certain that the color of the sky at the horizon and at the zenith is different. We are not able, however, to designate numerically the number of the different colors of which the total variety consists, because a line of demarcation can nowhere be drawn between the ending of one color and the beginning of the next. Therefore all the continuous diversities that we experience elude enumeration, although they do not escape co-ordination; for, to retain this illustration, every special color in the whole wealth of color in this sky has its own special place, and we should easily notice it if we were to undertake to transfer the color as it appears at a height of thirty degrees, as a spot, to a height of sixty or seventy degrees above the horizon. There it would be completely different from its surroundings and would not merge continuously with them. Thus the succession of colors from the horizon to the zenith represents an ordered variety, not a multiplicity consisting of members which can be counted, but which, on the other hand, are connected from beginning to end by a constant relationship. This, of course, is not the place to enter upon a discussion of the systematic construction of this whole theory of multiplicity. From what has been said one gets a sufficiently clear picture of how, by the

inclusion of one idea after another, more and more elements come into connection, through whose alternate union and mutual effect an increasing variety of relationships or special conditions arise, the determination of which is the mission of science. In the relatively simple case of geometry it has already been shown by recent investigations that at least sixteen different and entirely independent concepts are united to render the multiplicity of geometric phenomena. And the theory of combinations shows us at once an immense number of combinations of second, third, fourth, up to the sixteenth order, between these ideas; and what a variety, therefore, must be produced by the whole of a science so simple as geometry! And if we include in addition the idea of time, we pass from geometry to kinematics, the theory of motion, which proves to be considerably more varied than geometry.

All these sciences may be included under the idea of order, and hence may be termed in an inclusive manner the sciences of order. As regards their relations to the science groups previously formed, the most important thing to be said is that the most general theory of order is identical with the discipline which, ever since Aristotle, has received the name of logic. Aristotelian logic, to be sure, is only a very small part of the theory of order—that part, namely, having to do with the inclusion and exclusion of groups corresponding to certain definitions. Modern symbolic logic, or logistic (*Logistik*), as it has also been called, represents a more scientific and universal conception of the problems before us, but it has not yet arrived at the most elementary analysis of their concepts. This is due to the fact that symbolic logic has been developed from the side of mathematics, which is a still more complicated science, in that one thing after another has been thought out from the elements of mathematical concepts as presuppositions. This process of ab-

straction has already been carried quite far, and we now have a corresponding science which has been developed in recent years to an encouraging degree in a variety of directions. But the ultimate deductions have not been systematically completed, so that an accurate working out of the thoughts just outlined here is still lacking, and this first-principle foundation for all other sciences, which from the very nature of the case is all-important, is yet to be constructed. Thus much, however, is already known: that logic still passes as a postulate for all other sciences. For, as we know, there is no single science that does not consider as at least one of its aims the bringing of all its thought material into logical relationship—*i.e.*, the subjection of it to the laws of logic, or to the general theory of order.

The disciplines called arithmetic, or theory of numbers, and algebra, or theory of quantities, are still more special cases. For the general hypothesis which is made in algebra, that things belonging thereto can be added to or subtracted from one another, and furthermore that as quantities they can be subjected to measurement, is itself a limitation, since, as we have seen, there are also things which cannot be added or measured, and hence cannot be subjected to the other algebraical operations.

In connection with this description of the most general of all the sciences, we are at once confronted with a fact which is absolutely fundamental for the entire superstructure of the sciences, and with which we must, therefore, become as thoroughly acquainted as possible. We have seen that the most general concept of a thing may be defined by saying that that thing may be differentiated from all other things. This characteristic of differentiability is evidently a characteristic which is presupposed in any scientific problem. Whether we are examining chemical substances or search-

ing for the natural laws having to do with agriculture, we must always be able to differentiate the objects with which we are busying ourselves in order to be able to talk at all sensibly about them and to determine their natural laws. In other words, this means that those elements of an idea which we have found occurring as the *most general* ones we also encounter anew, owing to this very characteristic of universality, in connection with all the special ideas which we meet in any way or place in the more *special* departments of science. The most general sciences, therefore, will inform us concerning relationships which are not confined to these sciences themselves, but are found in all other sciences which arise through specialization from the more general applied ideas,—which treat ideas, therefore, that contain more constituent parts and more diversified ones than the ideas of the more general sciences.

The fact has already been emphasized that logic, for example, is a criterion for all the other sciences, and that its laws must be fulfilled before there can be any question of special laws in the other sciences. We also find that the quantity characteristics and the intensive variations parallel thereto that have no quantity characteristics occur in all the other sciences. Whether we have under consideration sociological or physical problems, we endeavor in each and every case to apply number and measure to them, and we think that we have made unusual progress in these sciences if we have succeeded in applying more general principles and methods of this kind. In the same way we apply to geometry the ideas which we have developed in arithmetic and algebra, and kinematics in its turn presupposes again all the ideas and relationships of geometry in order to be able to express thereupon its special laws.

Here, then, we have a natural law for the formation of

all the sciences. *The more general ideas and laws enter as regular component parts into all higher or more special sciences,* and there is no possibility at all of making any sort of scientific assertion in these more special or higher departments if the hypotheses are not fulfilled which the laws of the lower or more general sciences demand. We shall, therefore, be able to say that every higher science is divided into as many separate divisions as there are lower sciences to be found below it. The most complicated and highest of the sciences that we have considered—kinematics—will, therefore, have its algebraical and arithmetical and finally its logical side, for all the laws of the sciences just mentioned are already presupposed before one is able to set up the special kinematical laws. Hence, to express the matter in a purely scientific and technical manner, the variety of science, or the number of headings into which it falls, must become greater and greater the higher we ascend the pyramid of the sciences. This point of view will be of decisive importance to us, especially in connection with the later, more complicated sciences, in dividing and reviewing them.

We turn now to the second group of pure sciences, which treats of ideas that are lesser in range, but, on the other hand, are more diversified in content, than the ideas that have been richest in content heretofore, and which have been employed in the field of the sciences of order which we have just completed. These more diversified concepts are *space* and *time,* and it will be well, perhaps, to convince ourselves that both ideas are already of a very complex nature. We are accustomed to think of space in the following manner: we know that it has three dimensions, each independent of the other; that it therefore represents a threefold manifoldness; that it is in other ways continuous and without direction—*i.e.,* that it is alike in all directions. It is to a less

degree a matter of common knowledge that time represents a complex of quite a number of ideas. One can soon convince himself, however, that it certainly is not of an elementary nature, for it shares with space the characteristic of continuity; it is not, however, of three dimensions, but of one dimension—*i.e.,* in other words, one can pass from one point of time to another only in one way, and not, as in the case of space, in a threefold infinitude of different optional ways. Moreover, there belongs to time a characteristic which we do not encounter in connection with space, namely, the lack of symmetry. We differentiate the past from the future with absolute certainty, while in space it is entirely arbitrary what direction we call forward and back, or up and down, so long as we leave out of consideration other relationships not having to do with space. Moreover, time has the characteristic that, in spite of its one-dimensional nature, it never overlaps itself, there is never a point of time that belongs at the same time to an earlier or later time. These are all characteristics which can be expressed only with the help of simpler ideas, and which, therefore, make one realize how very composite and complex an idea is, taken in such a form from our experience.

In this connection we can only mention in a few words the fact that by means of the latest developments in physics, especially through certain optical experiments, science has been able to subject the ideas of space and time to a revision which has led to a peculiar synthesis. According to these developments, the details of which cannot be elaborated here, space and time are not to be thought of at all independently of each other; but, on the contrary, terrestrial occurrences are represented by a four-dimensional multiplicity, three dimensions of which belong to what has heretofore been called space and one to time. These dimensions, however,

are not mutually independent, for the definition of time in various places and that of space at various times condition each other in a special way. This points to a weighty general point of view, with which we shall soon be intimately confronted. We may ask ourselves whether the simple ideas upon which we gradually build up our system of ideas are of such a nature that each newly added idea comes as an additional degree of *higher* order to those which have been treated and employed previously. In that case there would exist among the various simple ideas a definite hierarchy, according to which, in the first place, are employed for the construction of a scientific system those general conceptions which have the most universal character, and then, of the other ideas, those which follow each other in this hierarchy should by degrees be sought out and employed for the construction. Up to the present this question has not been subjected to a close examination, and the possible answers thereto can therefore only be touched upon. It appears from what one has been able to see heretofore that a double relationship obtains. Single conceptions actually are built up one above the other in this hierarchical manner. This we see from the fact that after examining the question from all sides, there is indeed but one single definite sequence of sciences that permits of arrangement one above the other according to these principles. From this we must conclude that the new conceptions appearing in connection with the higher or more special sciences are of such a nature that they can appear only at this point, and, on the other hand, can play no rôle in the sphere of more general concept-building. On the contrary, we observe (and the relationship of space and time just described affords us an example of this) that when we arrive at a certain stratum of science formation it is not one single newly added conception which determines the

new science, but several. In this case, then, it would be the ideas of space and time which, since they are not independent of each other, cannot occur as steps one after the other independently of each other, but determine simultaneously the new stratum of thought and science.

In the observations which we shall now have to make regarding the sciences next in rank—the energetical—we shall find a similar case. The various types of energy appear as absolutely parallel conceptions which have no natural gradation as regards each other,—at least, none has been recognized and proved beyond possible question,—and which may be used, therefore, beside one another in whatever sequence one pleases. By their application to the classification of the sciences there result a number of special sciences, which are to be arranged one beside the other, but not one over another. With these hints, I must let the matter rest here, and only state in general that investigations like the above have not as yet been carried out in science from sufficiently general standpoints, so that it is frequently new territory, not yet investigated or worked, which must be entered. The uncertainty naturally belonging to such soil is lessened owing to the fact that we have already come to an understanding concerning the more general principles according to which the investigation is to be carried on. Therefore each single case that has not yet been more thoroughly examined may be clearly and definitely determined, thanks to these principles.

We now turn to the second great stratum of the sciences. The first we called the sciences of order, after their all-important, determining central idea, and the second we shall call the *energetical sciences,* because in this case the idea of energy is shown to be as general and determining as was order in the sciences mentioned heretofore. The sciences

which we shall discuss here are also called the physical or inorganic sciences; according to traditional division, they consist of mechanics, physics, and chemistry, and have to do with far more complex ideas than those with which the sciences of order had to deal. Within the sciences of order a cube is determined, so far as geometry is concerned, by the length of an edge; for the geometer there are not two or more different cubes, the measurement of whose sides is one centimeter. In physics, on the other hand, such a cube may turn out in a great variety of states: it may be different in density, in color, in temperature, in electric activity, etc., etc. Still greater variety occurs in connection with this idea in chemistry, where the cube may be formed not only out of more than a hundred thousand different chemical substances, but also out of an infinity of solutions and compounds made from them, and therefore it appears to the chemist each time as a different object.

We have already designated energy as the central idea within this sphere. This means that one can express all the variations of which we have just spoken within the whole sphere of physical sciences by certain statements regarding their ratio of energy. Since it is here a question of a relatively new thought formation which stood in a certain antithesis to those hitherto encountered, a few words concerning this matter are necessary. The field of the phenomena which have just been characterized is more commonly termed at present the field of material phenomena, or of phenomena occurring in matter, in which connection we understand matter to be the foundation, or what is permanent in the diversity, of physical phenomena. A general characteristic of concept formation has made itself felt in this connection, which we can also show, even if not quite as clearly, in the ideas of the sciences of order, but which is especially fa-

miliar to us in the field of which we are at present to treat. This is the idea of substance. We have already seen that our mental attainment is characterized essentially by the ability to remember, through the circumstance that the recurrence of an event affects us differently from its first occurrence. Owing to the function of memory, those which are repeated are endowed with the characteristic of familiarity, and thus, when we encounter new experiences of so well known character, we are placed in a position to feel at home or at ease with them; that is, the different component parts of the experiences are expected and presupposed by us, because we have impressed their connection upon our minds by the function of memory. Things which always recur together as regards space, such as are represented by the ideas of "apple" or "stone" or "tree," we are accustomed to consider as belonging together, and to call with special emphasis "objects" or "substances." Here it is a question of such formations as show a quality of stability; and therefore each time that we come in contact with them, a school of philosophy which denies so-called reality,—*i.e.,* the existence of things independent of ourselves,—owing to the circumstance that we can experience such things only in our consciousness, thinks itself justified in drawing the conclusion that these things exist only in our consciousness. I shall not occupy myself with a refutation of this opinion, but merely call attention to the fact that no one of the philosophers who share this opinion arranges his practical life correspondingly. Each of them, like the rest of mankind, actually demeans himself as a realist as regards the facts of life—*i.e.,* he recognizes in his practical attitude toward them that they have an independent existence in no wise influenced by his consciousness, and confines this spiritualistic theory of existence to his books and lectures. And there, too, we may

leave them, because it is my intention to deal only with realities—*i.e.,* with such things as have practical results, and especially with such as enable us to foretell the future with certainty.

The ideas of such substances are formed, as we have seen, automatically from our experiences through the function of memory. Our whole language, in its nouns, is full of the names of such substances, in which, however, the characteristic of stability changes within various limits. We encounter every possible and imaginable degree of stability, from things which exist only for a moment, but recur often with the same characteristics,—for example, lightning,—to things in which, within the memory of man, no lasting changeableness has ever been shown,—for example, sun and moon,—and therefore we always guard ourselves very carefully from connecting with the idea of a substance, or of a real thing, any kind of postulate concerning its "absolute" permanence. We can, of course, say to ourselves that substances in connection with which there can be shown a very high, or, so far as our memory serves us, even an absolute degree of invariability, will evidently play a predominantly important rôle in the formation of concepts. So we see that in the entire province of physical sciences mankind has continually sought for ideas by which such substances may be represented. And the whole thought of the period of scientific development which terminated with the middle of the last century rests essentially upon the fact that a concept of substance which had been developed up to that time—that of matter—was considered the most general and lasting. This idea of matter was formed quite rationally in the eighteenth century, men having co-ordinated with each related group of physical phenomena, determined by their similarity, a form of "matter" appertaining thereto. So there was, by the side

of the heavy, massive matter of ponderable substances, also matter in the form of warmth, of light, of electricity, of magnetism; and each of these terms serves to designate the existence of a definite kind of essence, or substance, to which a certain measure of durability and constancy has been attributed experimentally. Then, however, a certain degree of hesitancy regarding the completely ideal durability of many of these substances began to make itself felt. One can produce electric matter by rubbing glass and rosin; magnetic matter may be increased arbitrarily by rubbing unmagnetic steel rods; and in the same way one sees matter in the form of warmth arise in fire and disappear by absorption. We are confronted here by a twofold possibility: either one may extend the idea of matter in such a way that we may also include in it this process of appearance and disappearance, or we may limit the idea of matter to those things that we have experienced in which we are unable to show appearance or disappearance. Science has chosen the second route, and all imponderable forms of matter in which a process of appearing and disappearing can everywhere be shown have been eliminated from science. During the whole of the nineteenth century, since the time when Lavoisier proved—or, rather, since the publication of the postulate first promulgated by Lavoisier—that the sum of weighable substance remains constant under all circumstances, the idea of matter has been confined merely to these ponderable substances. By this process every other form of matter came into a false position, so that one did not know how to classify these entities, which, however, had not given up their rôle in either pure or technical science. One finally got around the difficulty by deciding to consider them hypothetically as motions. This, however, has not been carried out consistently. Until toward the end of the nineteenth century one read not infre-

quently of electric "fluidum," even though the expression electric "fluid" was avoided on account of a sort of linguistic feeling of embarrassment.

This un-coördinated and, so far as system is concerned, insufficient character of science has been fundamentally obviated by the discovery of the law of the conservation of energy. In his fundamental work, Julius Robert Mayer, the first discoverer of this law, undertook the single, definite task of finding by the side of weighable matter (whose constancy in all transformations, so far as weight and mass are concerned, had been fully confirmed by the physical and chemical investigations that had been made in the meantime) still another similar substance or matter which would include the imponderable part of natural realities. Mayer called these things *forces,* and hence sought, as he expressed it, to find a law which permitted of attributing to these forces that kind of uncreatability and indestructibility that had been noted in matter. His investigations at first led him into error. After he discovered that by the employment of mechanical action warmth was created, and, on the other hand, that in heat machines mechanical action can be attained through the employment of heat, he sought long in a wrong direction for the mechanical expression which could be proved equivalent to heat. Only after having endeavored in vain to provide the mechanical quantities, as he encountered them in text-books of physics as the measure of force, with this characteristic of uncreatability and indestructibility, did he finally arrive at the conclusion that the desired characteristic of mechanical *work* is due to the product of force and distance. With this discovery he revived an idea already expressed with utmost assurance and definiteness by Leibnitz upon a basis of purely mechanical considerations. Leibnitz had already shown that in the course of the mani-

fold changes of mechanical agencies into each other there always remains finally, in like constellation, in the ideal limiting-case, that which he called living force, the quantity which we now term *kinetic* or *motive energy*. Mayer's addition is the clear recognition of that which had found only short and rather hypothetical mention at the hands of Leibnitz, that in all cases where this law of conservation does not hold, another form of *force* (as Mayer called it), *energy* (as we call it), has taken the place of mechanical work or natural power, and this other form is, in general, *heat*. In the meantime Mayer extended this idea to all other forms of energy known at that time. We are indebted to him for the view that in addition to ponderable matter there was another sort of substance, an unweighable substance, which he called power, and which, like weighable matter, possesses the property of remaining quantitatively unchanged, whatever be the qualitative changes that it undergoes.

It is a well-known fact that this idea was arrived at independently by Joule, Helmholtz, and a few other investigators, and was further developed by them, and that down to the present it has led to the most definite and far-reaching theory concerning all physical phenomena that science has ever possessed. We now know that all physical phenomena, inclusive of chemistry, may be quite sufficiently and accurately defined and characterized as transformations of energy; so that, given the kind and amounts of the forms of energy that disappear and those that arise, every physical event is sufficiently characterized.

We thus see that the idea of substance, which was decisive for the origin of conceptions within the sphere of physical phenomena, assumed its most accurate and all-important form in the idea of energy. And it only remains to call attention in a few words to the fact that in this regard

energy has a more real meaning than ponderable matter, which had been looked upon from a physical point of view as the essential part of all phenomena. For the most recent development of physics has brought us to the point of view that mass can no longer be considered as something that is unchangeable under all circumstances, and that under present conditions we can no longer consider as absolutely correct either the fundamental mechanical law concerning the conservation of mass, or the law, regarded as equally fundamental, concerning the conservation of the quantity of motion or of the center of gravity. It is true of the greatest part of the phenomena of which we know. In a certain number of phenomena, on the other hand,—those, namely, in which radiation occurs as an energetical element,—further and more general formulation is necessary, and with its development science is now engaged. The history of this matter comes to a focus, however, in the fact that energy, as a matter of course, must be recognized in general and in particular as a substance. For, although in this last-mentioned great change in physics all other laws of conservation, and hence all other previous ideas of substance, have experienced upheavals and interruptions, no one has yet dared, even in the course of the most far-reaching speculations, to doubt the law of the conservation of energy. This is not due to any sentimental reasons; for the law regarding the conservation of energy is not much over half a century old, and therefore has by no means become such an almost ineradicable element of our mentality as had formerly been the case with the fundamental laws of mechanics at the hands of scientific people. The cause is, rather, that while those other substances, especially weight, mass, and the similar quantities that we have mentioned, extend their significance over quite a considerable proportion of science,—but

not, however, over the whole range of phenomena coming here into consideration,—energy, on the other hand, is the absolutely universal idea which finds its application in *every* physical phenomenon,—yes, as we have seen, in connection with every occurrence in general. To question the accuracy of the law of the conservation of energy would mean in fact to bring about a much more far-reaching upheaval in our previous methods of thought than to discuss the changeability of quantity (shown according to newest theories to be dependent on its rapidity) and other more circumscribed questions.

Now this is the reason why we in general term the great new field of science, which has been opened by specializing the previous more general ideas, the field of the *energetical sciences.* In regard to energy we know that it is a quantity taking part in all the phenomena of the entire field, and which, as regards all these phenomena, is subject to the law that to disappearing amounts of any forms of energy there always correspond like amounts of other forms of energy which arise simultaneously, so that the sum of all forms of energy remains constant. The question, then, how can one measure these various forms of energy so that their sum may be arrived at, may be answered by saying that it has been agreed upon to regard as equal those amounts of energy which arise from an amount of definite form of energy taken as unit (*motor energy* has been chosen for this purpose), or which change into this unit. This looks very much like a *petitio principii,* for if we call the amounts arising from the transformation alike, then the sum of the amounts of energy so measured must naturally be constant under all transformations. However, it is here a question not only of a formal determination of this kind, but of an actual natural law which arises from the following considera-

tion. Given three forms of energy, *A*, *B*, and *C*, assume that we next change the unit amount of *A* into *B* and define the amount obtained thereby as the unit of energy *B*. We then transform this unit into the third form *C*, and again define the amount thus got as the unit of the form of energy *C*. We can, in the third place, however, transform the unit amount from *A*, instead of passing by way of *B*, directly into the unit amount *C*, and the empirical law which thus comes to light reads that one gets in this direct way exactly the same amount of the energy *C* as one would have got through both steps from *A* to *B* and from *B* to *C*. If one adds still a fourth form of energy, *D*, there are not merely two different ways, but six of them, in which one may transform the unit amount *A* into the amount *D;* and experience also shows in this case that one always gets the same amount *D* from the unit *A* in whatsoever way the transformation may be undertaken. Hence can be formulated the general natural law that the amounts of any kind of energy that are got from any other kind are not only determined by this first and last form, but show themselves to be in no wise dependent on the intermediate forms nor on the multiplicity of ways in which the transformation results. This is the real content of the law of conservation of energy, and this content finds its shortest expression when one attributes to each form of energy its value in the way described in reference to the unit of an amount of energy taken as normal, and calculates with this value as with real quantities which can be added and divided without losing any part of their value on account of the manner of their arrangement or origin.

It is not necessary for the general observations which we have undertaken here to consider the different forms of energy which exist. It will be well, perhaps, only to say that the old division of physics into mechanics, acoustics, optics,

electricity, and heat can no longer be considered as logical. In the first place, acoustics is a part of mechanics, as has long been known, even though, of course, thermic phenomena do not play an unimportant rôle in it; then, on the other hand, optical phenomena have been recognized as a part of electromagnetic effects at a distance, and the most recent developments of this science even make mechanics appear as a part of electromagnetism, while at the same time a new electric theory of chemical action, at least in its initial phase, has been noticed. It would, therefore, appear at present as if we should be able to trace all other forms of energy back to electric—or, more exactly expressed, to electromagnetic—energy. However, development in this sense is only just in its most elementary stage, and therefore it cannot yet be stated with a sufficient degree of certainty whether the way that has just been pointed out, that would also conduce to the inward unity of the various forms of energy, can really be followed to the end. It is not impossible that the position of the electric theory of all physical phenomena in the course of half a century or so will be similar to that of the mechanical theory of all physical phenomena at present, namely, that it will demonstrate itself ultimately as unfeasible.

Finally, we should not fail to mention just here that the various forms of energy are not to be looked upon as ideas placed in a higher order, but rather as collateral, new, complex ideas in this field. For this can be assigned only the partial reason that the forms of energy that have been traditionally taken together in physics show a closer relationship to each other than to chemical energy, which, owing to the excessive variety of its phenomena, has for several centuries developed as a special science as compared with physical phenomena, and shows also certain fundamental new variations. Whereas, for example, it is a matter of indiffer-

ence in electrostatics whether a conductor of any definite form—for instance, in the form of a ball which is a meter in diameter—be made of tin or gold, of iron or lead (for electrostatic capacity depends only upon the form and environment of the conductor, not upon its special nature), for the chemist the various balls just mentioned are absolutely different objects, and in his eyes are endowed with the characteristics of mutual untransformability and lasting difference. For as regards chemistry before the present time, the law of the conservation of elements has been as valid as the law of the conservation of volume is for mechanics. But the same investigations that cause the law regarding the conservation of volume to be viewed only from what must be admitted frequently as a complicated special case of a general law, have also led us to view the law of the conservation of elements as a very general law whose conditions, so far as the occurrences known down to a decade ago were concerned, had always been fulfilled, while the facts which have been observed recently in connection with radioactive substances lead to the establishment of exceptions to this law. We are, therefore, led to conclude that there are some more general laws, as special cases of which these particular laws of conservation appear, which, however, under certain conditions and hypotheses, also permit a non-conservation, possibly a mutual transformation, of such quasi-substantial qualities.

If we now undertake to prove the proposition just laid down, that the laws of the more general sciences everywhere and in every detail must be true of the laws of the higher and more special sciences, we are able to convince ourselves readily that it must be so throughout. That we cannot treat of all the physical and energetical sciences without logic is a statement which is so trivial that one almost hesitates to

express it; however, it must be mentioned here for the sake of completeness, and also for the reason that the position of logic as the most general of all sciences—more general, in fact, than mathematics—is by no means commonly known and recognized, although for a decade I have pleaded for this point of view, which is so fundamental as far as method is concerned. It is also just as much a matter of common knowledge with us that we have to apply mathematics and geometry, and finally kinematics, to all the phenomena of energetical happenings. It is well known to every one acquainted with the history of these sciences, that especially the introduction of mathematical and geometrical methods into the treatment of physical phenomena has brought with it enormous progress in our comprehension and treatment of them. Does not quite an appreciable proportion of our highly developed technical knowledge of to-day rest upon the fact that we have learned to apply number and measure to the various physical phenomena, and hence to foresee the results of certain constructions and combinations, so that they may be exactly determined, not only in respect to kind, but also as regards amount? The construction of all modern machinery rests in fact, as we have seen, upon a knowledge of mechanics and thermodynamics. Electrotechnics, too, which has begun to transform our outward life so successfully, and whose influence upon this transformation is by no means terminated as yet, has been completely developed upon a mathematical-geometrical basis laid so successfully and deeply by the geniuses of electrical theory, from Ohm through Faraday and Maxwell to Hertz and the investigators of to-day. In this very department of the physical sciences, more clearly than in any other province of knowledge, is shown the extraordinary assistance that the systematic introduction of the earlier and more general sciences

has brought with it into the investigation of the higher and more special sciences. As a most impressive example of recent times physical chemistry may be mentioned, which also rests upon this kind of application of the more general mathematical and physical concept formations to the phenomena of chemistry, through the operation of which problems have been solved in a few years which the usual method of investigation in vogue up to that time, that clung more to the immediate phenomenon and took no consideration of any further means of assistance, could not have touched in a hundred years.

We have now noted what is most essential regarding this second stratum of the sciences, and we have yet to call attention to the fact that the variety in this field may easily be surveyed synthetically by means of conjoining the various kinds of energy. Within the range of all the physical sciences the legitimacy of each single kind of energy must first be established. Then each one of these kinds of energy must be combined binarily with each of the others, whereby new localities result from their reciprocal action. Thus, for example, the characteristics of vapors have been investigated, on the one hand, by the theory of heat; and, on the other hand, one could apply to them the mechanical laws studied in gases. By means of the combination of the laws of mechanics and heat thermodynamics arose, the science which has taught us the nature of the agent so important to steam-engines, upon which the whole enormous development of the corresponding technical science of the present rests. To the binary combinations of two forms of energy the ternary must be added, and so forth, until all the combinations possible have been exhaustively worked over. By means of this seemingly outward manipulation, but one which is in fact fundamentally scientific, not only a complete

diagram of all the possible and conceivable disciplines of physical science can be constructed, but one may even predict to a considerable extent what forms the special laws will assume in the various columns of this table. Moreover, a diagram of this kind makes possible the immediate drawing of conclusions in case a form of energy is discovered which has not been previously observed. One has to bring this new kind of energy, *X*, as a new member into the whole calculation or combination with the pre-existing kinds of energy, and to form again as a consequence, after having determined their laws by the combination of *X* with the energies *A*, *B*, *C*, etc., a group of binary, and later, as has been described, a group of ternary and of more highly complicated fields, by working through which one may be certain of exhausting methodically all the physical disciplines that permit of survey down to that moment. Such a situation is so highly desirable and valuable that under all circumstances we should do everything possible lying within the range of science to attain to it.

An especially instructive example of this scientific process of extension by means of the inclusion of the ideas already derived from the earlier and more general sciences is presented by the most recent development of chemistry in respect also to the application of the ideas of time and space to special chemical problems. The incorporation of the time idea in chemical phenomena led to the great field of *chemical kinetics*, which has borne fruits so abundant, and in which, in spite of the short duration of its previous scientific existence, progress so noteworthy has been made, both from a theoretical and from a technical standpoint. It need only be mentioned that it was in this field alone that the phenomenon of catalysis, which had been known for a century, was able to attain a logical explanation. As to the

application of the idea of space to chemical phenomena, we need only mention stereochemistry, which at the present time also represents a science that has arisen only in the last decade, but which already has a wide range of application, and in which the idea of the multiplicity of space has been successfully applied for clearing up chemical diversities, especially isometrical relationships. Here, too, it has been possible, by carrying out logically the basic idea, to make a great number of chemical prophecies which later experimental investigation has confirmed down to the smallest details.

We now turn to the last group of sciences, whose ideas are the most complicated and therefore the smallest in scope but richest in content. This group arises from the fact that to the ideas that we have thus far arrived at in the field of order and energy, that of life is added. By phenomena of life we understand very definite transformations of energy by virtue of which the objects in question—the living beings—accomplish a continuous transformation of free energy, consumed either in the form of chemical food, as in the case of animals, or in the form of the radioactive energy of the sun, as in the case of plants. Over and above this continuous or stationary transformation of energy they are distinguished, moreover, by the capacity for reproduction—*i.e.,* the production of new similar types, by means of which individual mortality of single members has been transformed through time and space into a disproportionately longer continuation of the *species,* the totality of similar individuals. Thus in connection with the scientific examination of life we have to presuppose for its ideological comprehension and definition the totality of the sciences of order and the entirety of the physicochemical or energetical sciences. In so far, therefore, we shall have to say that every living being is an

energetical type, and that all the laws that we have found for such a being must find their legitimate application to living beings. We shall have to say, furthermore, that a new conception has appeared here,—that of life,—which is characterized by stationary transformation of energy as well as by the capacity of reproduction, and concerning which we cannot maintain that it can be completely defined by general physicochemical laws. For we are quite in a position to differentiate experimentally living beings from those without life, and this fact alone suffices to prove that new relationships have appeared in connection with this narrow group of things, the ideological comprehension of which gives the scientific definition of life. Hence we shall have to consider every living being as a physicochemical object, in so far as nothing can occur in this object that does not take place within the compass of the energetical laws. But we shall have to consider animals as formations of a special kind in so far as certain peculiarities belong to them which are by no means present in all energetical objects, and which, therefore, render necessary special treatment and scientific discussion of them.

The science of living beings we term in general *biology*, and we divide this whole discipline into single groups according to the special kind of life activity, and, at the same time, according to the increasing intricacy of the entire organization of the living being. The most general characteristics and relationships which occur in all living beings, and take on a one-sided and specific development only in the case of certain ones, according to special forms and purposes, we treat in the form of a whole science bearing the name of *physiology*. In the very first place, it is a question here of physicochemical conformity to law. The special characteristic of physicochemical happening in the living being must be shown here

in detail and explained experimentally; and, inversely, the physicochemical hypotheses must be found regarding the activity of all specific happenings in living beings, their single functions. Thus the principles of division which were determining for the energetical sciences make themselves felt also as secondary reasons for division in physiology, and the corresponding groups have also been formed already in this science, such as electrophysiology, mechanical physiology, chemical physiology, etc.

A special apparatus in connection with which new kinds of phenomena arise, which have led, therefore, to new formations of concepts, is not found in all living beings, but only in those in which a division of functions has taken place, and hence in which the necessity exists for uniting these divided functions for the purpose of harmonious and suitable working. This is the nervous system, which in the case of the more highly developed animals is grouped about a central organ which, as we ascend the scale, is formed in a more and more complicated and abundant way, until it reaches its highest development in man. The special relations that occur in the function of this central organ are what form the subject of this higher and more special science of life, which, from the name for the totality of this function in man, we call *psychology*. Here, also, we shall have the same things to say about general biology, namely, that for the investigation of psychological relationships in lower and in higher living beings, finally in man, the knowledge and efficacy of physicochemical as well as of general biological laws must in all cases be presupposed, and that here it is only a question of specializing the mode of operation of these laws according to the special conditions under which, in the first place, nervous phenomena—in a narrower sense, psychic phenomena—occur. Since these psychic phenomena also presup-

pose energetical happenings, even occurrences in connection with ponderable substances which are endowed with chemical energy, we must consider them of course as energetical occurrences, and the old problem of the connection between mind and matter attains a satisfying systematic solution in the light of the general system of science here described. Psychic phenomena, in the next place, must be considered as resting upon a definite energetical basis. Within this limit, however, they are specialized by peculiarities connected with the function of the nerve tracks and central organs.

Finally, an uppermost layer of this pyramid of sciences is formed by those facts and relationships which have developed in *man,* in contradistinction to all other animals, and which form that which we specifically call human *civilization.* This science is usually designated by the improper name of *sociology.* The name is due to the fact that man, even in the very early stages of his development, has unquestionably been a social being, so that, for much the greater part, specifically human culture has shown itself to be the culture of groups of people living together socially and busying themselves in common. This special nature of human culture, however, is relatively a secondary phenomenon; and it is, moreover, not entirely general, for certain cultural performances have been, and can in the future be, accomplished by a single individual. Thus, socializing mankind is an important phenomenon in this field; indeed, it is one of the most important, but not the characteristic and universal one. I proposed, therefore, a long while ago to call the field in question the science of civilization, or culturology (*Kulturologie*). And though it is not my opinion that anything of very great importance for science depends upon the acceptance or refusal of this proposal, I think, nevertheless, that in the present indefinite situation in which the science of civili-

zation, or sociology, finds itself as regards its general principles and its place within the field of the other sciences as compared with the generality of them, a sharper emphasis of this kind on the essential feature of this new science might be of some benefit.

To culturology, or the science of civilization, numerous sciences belong which we are accustomed to include under the name of *mental sciences,* the retrocedent nature of which, to express it in terms of method, we have already discussed and explained above. Law and language, administration and agriculture, industry and science, religion and art, are all merely different forms of activity proper to the general cultural work of humanity. Any investigation of them must, therefore, take the direction of applying the laws of the corresponding occurrences from what the historical knowledge of earlier phases and the anthropological examination of contemporaneous phases of less developed peoples and of other groups of human culture has placed at our disposal, in order to determine thereby the present *niveau* of a given field of culture and its prospective development. What we call politics in its wider sense, not only the relations of one state to another, but the general technique of the administration of common possessions and the education of coming generations for the corresponding activities of the community,—this wider kind of politics, including the politics of civilization, shows itself under this aspect to be the field of application for scientific culturology or sociology; and, speaking ideally, through the development of this latter science in the future politics should be formed and conducted with the same certainty and precision with which we build at present an iron bridge or a station and understand how to direct an electrical or steam plant of so many thousand horsepower and keep it going.

Culturology, appearing thus as the topmost course of the pyramid of the sciences, shows itself from the point of view of method also to be the most diverse and many-sided of the sciences. For all of the more general sciences, logic, mathematics, geometry, and kinematics, as well as all the energetical sciences, and finally general physiology and psychology, have each its influence upon the formation of culturological ideas. A sure mastery of at least the fundamental principles of all the sciences that I have just mentioned is therefore a necessary presupposition for the scientific mastery of culturological problems. If one considers that science of the twentieth century, even, is far from enjoying a sufficient development of them, especially of the biological sciences, and that the application of the sciences of order to cultural science has already made some progress (especially in the sphere of political economy and in its technical application—statistics), one realizes that the application of the energetical sciences to the science of culture has almost been mapped out provisorily in its fundamentals. Still less can there be any question of a rational general application of biological theories to the science of culture, in spite of the fact that tentative efforts in that direction have already been made.

Thus one sees with what an enormous problem we are confronted, one that is scarcely to be compassed with our present resources; and it is quite comprehensible if the workings of previous mental sciences, which have not been able to await the systematic development of concept formation in the lower sciences that are so necessary for any rational treatment, leave so very much to be desired at the present time on the side of scientific method. In the field of culturology it is still almost universally a question of the technical period of science, for nearly all of the special cul-

turological sciences are at present only in the stage of their own development determined by practical necessity. In this connection, I need only remind you of the present condition of jurisprudence, which shows precisely the characteristic forms of development which have been outlined here. Mankind has not been able to wait until the twenty-first or twenty-second century, at which time it will perhaps be in possession of a pure or methodic culturology, to bring its affairs to such order that it might keep the body politic alive and capable of functioning. In the very same way, mankind has not been able to await the development of physiological chemistry in order to procure and prepare the food inevitably demanded day by day in order to preserve life. Thus, jurisprudence of the present is nothing but a most unsystematic sum of all previous attempts made by especially endowed empiricists to preserve the social and scientific order of a community of persons. The idea is very far from the mind of the jurists of the present, that all the problems relating to jurisprudence must first be illuminated with the fundamental principles of the physical or energetical sciences in order to place it upon an exact basis. If, however, one considers, for example, how exceedingly irrational our present criminal laws and penological procedure are, based almost entirely upon imprisonment, how by this process society is neither freed permanently from the evil-doer, nor is the latter placed under conditions in which he gives up as far as possible his antisocial habits and replaces them with social ones, one realizes what an enormous amount of work yet remains to be done in this field before we shall be able to speak of a real, scientific theory of law.

The same thing can be said of *language,* which represents the most important social means of communication, and whose duty it is to render the mental concepts of individual

persons accessible to other members of society, and then, by means of written characters, to insure their effectiveness for posterity over and beyond the life of their creator. This conception of language as a means of communication, and the criticism of language resulting therefrom, according to the standpoint of its *technical* adaptability to the exact, unequivocal, and sufficiently complete expression of the ideas formed by each individual, as well as to the transference of ideas from one individual to others,—this conception of language, I say, does not yet play the slightest rôle in the science of language. Instead of properly envisaging what is essential in phenomena of this kind,—the ideas, their co-ordination and system,—and making them the subject of scientific work, linguistics had heretofore limited itself almost exclusively to the most unimportant and least necessary of the whole phenomenon, namely, to the forms, in sounds and characters, which have been co-ordinated with the preconceived ideas. The extraordinary diversity of the various languages certainly shows clearly how very unimportant the special forms employed by single groups of people are for what is essential in language—social intercourse. Nevertheless, what has heretofore been called linguistic science confines itself almost entirely to the investigation of the nature, or at most to the investigation of the slow changes which these accidentally co-ordinated characters have undergone; while practically no attention at all has been given to an investigation of concept formation, to a system of the ideas themselves, to the question as to what classes of ideas there are, in what way simple and compound concepts react upon one another when combined—in a word, to the problem of the science of concept formation. So we need not be surprised that the fact is known to but few people that at the present time the technical problem of an artificial lan-

guage which is more complete than any natural one has already been solved. It is of special interest to note that the possibility of such a thing is most emphatically denied by the representatives of previous pseudolinguistic science, the philologists, though facts for years have proved the contrary.

If we glance back over the observations thus far made, we become aware that all of the sciences, taken together, represent an absolutely coherent complex, ascending from the simplest to the most involved, but exhibiting at every point the same course and the same character of progress, and consequently give no occasion at all for any delimitation of the frontiers of opposing fields as regards one another. It is therefore absolutely incorrect to separate, as is often done, the entire field of human sciences into two groups which have little or nothing to do with each other, and whose functions are fundamentally different. Regarding the group of the natural sciences there is complete agreement. Other sciences, however, which were formerly termed mental sciences, were set over against them. Afterward one necessarily became convinced that the natural sciences, too,—for example, psychology,—had to do with mind, and that mind, therefore, was no special distinctive mark of this other department of knowledge. Then it was thought that the science of civilization must be placed in contrast with natural science, but it soon became evident that cultural phenomena form a group (and indeed the highest) of natural phenomena. It is unreasonable and impracticable to consider the activity of man in his surroundings as "unnatural," as compared with the activity of animals and plants. Finally, the sciences of this special group were called *sciences of volition,* because they rest upon the activity of the human will. This difference is not practicable, either, for without the

corresponding impulses of the will, which have been prompted by the exigencies of existence, no one would have busied himself either with the theoretical or with the applied sciences.

So there remains in fact no possibility of making an essential distinction, and only the historical difference exists that the treatment of the higher sciences heretofore has been largely carried on with inadequate means and without any information as to the real aim of all sciences, namely, the ability to predict. It is true that from this situation a contradiction has arisen which is destined, however, to disappear and will disappear all the more rapidly and all the more surely in proportion as the scientists in all the various fields become aware of the unalterable unity of all science. This unity of science leads us also to a great central problem, for the solution of which the representatives and incumbents of all the various sciences must co-operate.

This problem is to establish *a systematic inventory of all human ideas* upon the basis of the fundamental relations of increasing multiplicity and complexity that have just been explained in proportion to decrease in compass. Our preceding analysis of all knowledge has led us to see that some of these ideas, like those of order, of energy, and of life, stand out with especial clearness from the entire range of thought. But these ideas are all of a complex nature, and it is an inevitable necessity for the sure handling of the entire scheme of all the sciences that one should separate these very important collective ideas into their elements and arrange these elements in corresponding natural groups according to similarity and reciprocal efficiency.

This is a work the necessity of which was clearly recognized even by Leibnitz. We have from his pen numerous discussions of the extraordinary advantages which the hu-

man mind could derive from such an inventory of all its material for thought. But I am not aware that Leibnitz ever made the attempt to draw up a table of elementary concepts and to sketch, even schematically, the laws of their mutual effect in the formation of new ideas. I myself have been working on this problem for ten years, without, however, having made up to the present moment so extensive progress that I could give a consistent presentation of the entire matter.

In the course of these labors, however, certain points have been brought out as well as could be wished. There is, in the first place, the process of differentiating simple concepts from more complex ones. We recall the fundamental relationship between the content and the compass of the various ideas, and are enabled to establish upon it a means of defining the elementary notions. When we consider any idea and vary it, seeking out some nearly related one, the scope of this related idea will show itself to be either greater or smaller than that of the original idea. If it has become smaller, then the related idea is of a more complex nature than the original idea, and we have undertaken a synthesis instead of an analysis. If, on the other hand, its scope has become greater, we have simplified the idea, it has become more elementary. We can apply the same process to this simplified idea. If we finally reach an idea which cannot increase any more in scope by any form of change, we have arrived at an idea which may be regarded, at least provisionally, as elementary. Since it resists further analysis, it is entitled to a place in the table of elementary concepts.

This process, as one sees, is extraordinarily similar to the process of chemical analysis. In it, too, one proceeds by first subjecting a substance whose nature, whether it be elementary or compound, has not yet been established, to chem-

ical influences—*i.e.*, one endeavors to transform it into another substance with other characteristics. If a single second substance with increased weight arises from the substance submitted to us, and if, under all the conditions under which it is subjected to chemical transformations, some other sort of substance always arises whose weight is greater than that of the original substance, then we know that we have to do with an elementary substance. If, however, the substance can be transformed into others, each of which weighs less, or only one of which weighs less than the original substance, then we know that we have to do with a compound substance. If we subject the product of less weight thus arrived at to similar transforming influences, we can establish in its case also whether it is of an elementary or compound nature. In other words, under the supposition that a substance is compound, we treat it from every possible side with the agents by which chemical transformation is brought about, and observe whether it increases or decreases in weight, and if we have a substance which under all circumstances only increases in weight or keeps its weight unchanged, we have proved its elementary nature sufficiently well.

In this way the scientists who have chosen as their field of labor the investigation of the total problem of science will have to begin by examining all concepts as to their simple or compound nature, without any reference to any other relationships. From these results is, then, to be arranged a preliminary table of simple ideas which have been found thus purely empirically. These elementary ideas are to be pronounced elementary until their complexity is established, just as is the case with the elements in chemistry. According to the generally accepted definition, an element is really not an unanalyzable substance, but a substance which has not yet been reduced. In the same way we can say that an elemen-

tary idea is not an unanalyzable one, but an idea which has not yet been analyzed.

My previous work on the arrangement of a table of concept elements like this has shown me that these elements may be divided into two large groups of which passing mention was made earlier in our discussion. On the one hand we have the group of *substances* or *objects* or *things,* or whatever else we wish to call them, the group of those concepts which represent entities existing in themselves, which we always find recurring in the same way in the range of our experience, and which have, as regards time, an unchangeable or at least only slightly changeable nature. By the side of this group still another group of quite essential ideas is found, which we term ideas of *correlation* or of *relation* or of *reciprocal action.* They, too, represent quite definite experiences, but they refer regularly to two or more ideas of the first kind, and are the material by means of which the connection between isolated substances or things is brought about.

We realize at once that the psychophysical function of memory leads first to ideas of the first kind. Those elements of experience (since we are speaking here of elementary ideas) which always affect us in the same manner take on, then, the form of these substances or objects in our consciousness, and independently of ourselves assume this character of real existence which we ascribe to our external world. So long as mental functions are confined to the formation of such concepts of objects or substances, real thinking is impossible, since each of these concepts leads its own isolated existence and can in no wise come into connection with the others. Just here the experimental fact is added, that we never experience such concept elements in isolated form, but in coherent complexes which even as such

are felt to be units whose division into elementary component parts follows only by a considerable effort of the mind, for which a high degree of maturity and independence of judgment is necessary. This results from the fact of reciprocal connection, of the *relation* of substances to one another.

Thus these mental relations in the form of space or of time, or, to express it in general terms, of function, between the different concept elements of substantial nature, form quite an essential part of our total experience. The determination of such relationships between substances on the conceptual side has at least as much importance for our entire mental activity as the formation of the idea of the substance itself. The *association of ideas* which has been characterized and studied for a long time by psychologists is only a relatively narrow expression for this general function of relationship which is stamped upon our mind by the nature of its experiences. It represents, however, the best-known part of general ideas of relationship and permits us also to see the circumstances through which these relative ideas have been formed alongside of the ideas of substance. Such ideas of relationship, for example, are "by the side of" or "above" one another in space, "earlier" and "later" in time, and a number of others, all of which may be recognized by the fact that they never refer to a single object, but invariably bring two or more objects of different kinds into mutual relationship.

We recall from our preliminary description of all science that the idea of group, namely, the relation or connection between objects of like nature, appeared at the very beginning of our formation of ideas, and proved even then to be that process by means of which a mutual relationship arose from ideas of objects that until then had been disconnected, and with it also came the possibility of establishing natural

laws. The unconscious work of language, too, has clearly differentiated these two kinds of ideas: the object-ideas are characterized chiefly by nouns, but also by adjectives and other words, while the ideas of relationship are expressed chiefly by verbs. But since language, as has been mentioned, has arisen unconsciously—*i.e.*, without a clear consciousness of purpose or aim—the two great classes just referred to are by no means sharply distinguished from each other. For surely freedom in usage has given us on almost all occasions the possibility of making a verb of a noun, and, inversely, of considering in a formal way every verb as a substance-idea—*i.e.*, as a noun. But in such matters it is only a question of formal resemblance to the other group, whereas upon real analysis of the content of the idea connected with the words in question, their character as objects or as relationships can almost always be determined without difficulty.

Labors of this kind, which presuppose and demand quite a thorough knowledge of concept formation in all the sciences, represent now what I consider as the real rational task for a future philosophy, and one which will be useful—yes, indispensable—to mankind. According to this view, philosophy would be the science which is occupied with the sciences as a whole in reference to their mutual relations, their structure, and their circumstances. It has the practical mission, on the one hand, of predicting those fields of knowledge which have not been subjected as yet to any systematic treatment or to treatment of any kind, and, on the other hand, of rendering the existing fields of knowledge capable of easier advancement and better arrangement through the proof of systematic and methodical relationships to other sciences. By the cultivation of this new philosophy it will then be possible to organize and improve all functions of science which at present are so imperfect. The present procedure reminds

one of the growth of a primeval forest, where every single tree develops on its own account and by its own strength, as well as it can, and so far as it finds light and air. Under such circumstances splendid individual giants may grow, but only at the expense of numerous other trees which under other circumstances could have developed luxuriantly and beautifully, but which suffocate here under the shadow of the giant. Future science is more to be likened, therefore, to a logically cultivated forest in which every tree stands in its own place, and each, in proportion to its value, receives generous attention. To employ another figure, we still stand in our present attitude toward science as men stood toward the problem of economics when men were only huntsmen, and when the acquisition of prey, and hence of food, was essentially a matter of accident and of special personal skill. In our treatment of the sciences we wish to pass out of this primitive condition into a condition which may be compared to that of men occupied with agriculture, by cultivating regularly scientific progress. Owing to the fact that we prepare the ground suitably and arrange the conditions of development as favorably as possible, we shall gain, in the place of the accidental discoveries, which were at times quite abundant, but frequently extraordinarily scanty and insufficient, a steady harvest which, to be sure, is not entirely independent of the contingencies of external climatic conditions,—in the present case, of the multiplicity of political and economical conditions among men,—but which produces nevertheless, with slight variations, year in and year out, a regular, recurrent harvest and assures therefore a rational and careful collective science of humanity in this greatest and most important field of its entire mental activity.

An attentive reader has perhaps missed two things in this examination of all the sciences. First, a thorough considera-

tion of the applied sciences which are the mother earth out of which the general sciences have sprung. Furthermore, one may have noted the complete non-consideration of a discipline which is claiming at present an extraordinarily important place in our highest educational institutions, the universities, and the importance of which is being emphasized in a very lively way on many sides—namely, *history*.

As far as the first matter is concerned, it can be disposed of quickly and easily. One readily sees *that every applied discipline has its center of gravity in one of the general sciences*. Thus there is, for example, an exceedingly extensive and important applied science—astronomy. This had its center of gravity until half a century ago wholly in mechanics, for all astronomical phenomena which were then observed and which were essentially confined to the determination of the positions of moving stars, and of the energy of gravitation by which they are held together in single groups, like the solar system, for example. With the exact recognition of the nature of these two kinds of energy, begun by the investigations of the sixteenth century and terminated fundamentally by Newton, this astronomy of position became an essentially completed science, in the case of which, to be sure, there was refinement and inner development, but no further extension on the side of ideas. In the last half-century, however, a new and extraordinarily far-reaching auxiliary means has been introduced into astronomy through the discovery of spectrum analysis, by means of which other fields of the energetical sciences, especially chemistry, have taken up their rôle in the development of astronomy. In this connection the natural relation makes itself felt, that geometromechanical astronomy is a necessary presupposition for the investigation of astrophysics and astrochemistry. One must, of course, be sufficiently informed as to the gen-

eral questions of position and motion before one can attack these more intricate problems.

Thus, from this example we see how at first an external thing by its striking character and its technical importance (that of astronomy lies in its application for getting our bearings upon the surface of the earth, especially on the sea and in the desert) takes first those sciences into its service whose development has proceeded sufficiently far for the study and explanation of fundamental phenomena. In its further development it makes use of all the other sciences that can be applied to the existing relationships, and leaves out of consideration those sciences for which there are no possibilities of relationship. The whole development through which astronomy has passed rests upon the fact that the only news that comes to us from the stars is transmitted by light. Only the relatively few celestial bodies—namely, the planets, moons, and the sun—whose demonstrable field of gravity reaches to the earth and influences its movements, show in addition the influence of the energy of gravitation. The entire sphere of the fixed stars, of cosmic nebulæ, and of other formations in the universe is so distant from the earth that any effect of its fields of gravitation is in no way demonstrable; in the case of these there remains only radiance, therefore, by which any energetical communication whatever takes place with the earth and its inhabitants. From this fact it may be concluded on general principles that only that which light can tell us can be known by us about the stars, and that since no other form of energy travels from the stars to the earth, it is absolutely impossible to learn anything about other energetical conditions of the stars. Thus, for example, we are thrown entirely upon conjecture as to how biological processes may take place on Mars or Jupiter, for instance, the confirmation or non-confirmation of which is

absolutely without importance to the inhabitants of the earth in so far as there simply does not exist any energetical relationship between the eventual characteristics of the neighboring planets and those of the earth. According to general principles, one may imagine that the use of optical information from the planets may be developed so far that details of biological problems might also be studied, but obviously even in that case the possibility will be considered that other forms of energy, hitherto unknown to us, may be transmitted from star to star, and that we shall be able, if we become acquainted with such forms of energy, to deduce from them corresponding information, just as we now derive all the information that we receive from the stars from the energy of light.

Somewhat different from astronomy as an applied science are the technical sciences proper. Now, while astronomy is busied with the study of existing objects and makes use of their characteristics as basis for their application without being able to influence and change them in any way, in the sphere of technical sciences we have to do with objects and processes upon whose ordering in time and space and upon whose reciprocal action we are enabled to exercise considerable influence. We use this influence, then, to direct natural processes in such a way as may seem at all advantageous or desirable to us. Man's mastery of nature means nothing more than that he takes possession in an increasing measure of natural energies and learns with increasing skill to exploit them for his interests. At first we see how in regular succession the energies best known to man and most familiar to him—namely, other men's capacity for work—are put to use. This has found expression especially in slavery, which was general in antiquity and at present is being relegated more and more to those regions that are still in a stage of

barbarism. Then the more difficult problem was solved—the employment of the capacity for work in animals for *human* needs. The more recent phase of this general advancement consists finally in the fact that for not much more than a hundred years—but then, however, in rapidly increasing measure—*inorganic energies* have been placed in the service of mankind. This has been achieved down to the present day chiefly by means of fossil coal. But in most recent times it has become possible through the development of electrical engineering to harness the natural powers of water and to place them in the service of human labor, so that they are beginning to supplant the chemical energy of coal in an increasing measure. For fossil coal is not a possession that is being produced continuously and formed anew each year on the earth in proportion as it is consumed by mankind, but it is like an unexpected and unforeseen inheritance which has fallen into the hands of mankind, and which will also be exhausted at a date not remote. All the improvements of technical science which are directed toward a saving in the consumption of coal, or which render possible the exploitation of coal regions which were inaccessible to technical science in the past, can, after all, only postpone but not prevent the complete consumption of the coal supply. And if this accidental inheritance is exhausted, mankind will be forced to put to use that portion of the regular supply of energy—namely, the ever present solar radiation—which it needs for the furtherance of its civilization. Natural water-power represents an energy of this type for raising water by the influence of the sun's rays, and the condensation of vapor on the highest points of the earth represents a continuous process which will not change essentially so long as the conditions of life on the earth remain adapted to the human race.

The fact must, of course, be taken into consideration, that through this very process of running ice and water down from the highest peaks of the earth there results a gradual wearing away of these summits and a diminution of their height, so that upon closer analysis this form of energy is also one which is slowly diminishing. We shall, therefore, have to consider as an ideal solution of the problem some form or other of mechanical contrivance by means of which the rays of the sun may be caught up directly and transformed into other kinds of energy. Technical science, for example, which a few years ago, when the question came into prominence of there being a possible lack of latent nitrogen for producing food for mankind, at once put an end to this deficiency by developing theoretical and sweeping methods for binding the nitrogen of the air until it was rendered serviceable, also envisages such a task with the quiet assurance that it will not merely be solved when, owing to the consumption of the last piece of coal, mankind finds itself face to face with the bitter necessity of a solution, but that the solution will have been reached long before the last treasures of coal have been subjected to exploitation.

As may be seen at once, the problems here in question in connection with procuring primary energy for human purposes are grouped around the energetical sciences. Physics, especially the theory of heat and of mechanics on the one hand, and chemistry in the form of the theory of chemical energy on the other hand, are the basic sciences the theoretical or general mastery of which is a prerequisite for successful technical development. Other technical sciences have other theoretical sciences as a nucleus. For medicine, for example, it is physiology, especially that of man. In more recent times psychology has also been coming increasingly into prominence, and advances in it are rendering possible a

much more sure and successful treatment of mental disorders, anomalies, and defects. The future activities of both sciences will place within our reach in time to come the attainment of a healthier, stronger, and more capable progeny. Thus we can find for each technical science the sphere in which its theoretical foundations are laid and are being developed without reference to immediate application.

The science of civilization is especially fertile in that kind of applied disciplines (indeed, up to the present it has largely consisted of them) in which the theory of law, the theory of the state, education, and finally the whole organization of science, belong as technical branches. Since all the other sciences converge in the science of civilization, we see how extremely diversified this discipline must be in a theoretical as well as in an applied sense, and we see, for example, that certain disciplines, even, which according to previous belief stood outside of science, like ethics, must form a necessary and regular constituent part of sociology. For, from this point of view, ethics also is shown to be an applied science. It is the theory of the way in which and the content to which the individual must limit and direct the activities of his will in order to mold his own life in keeping with his own volition as far as possible, but yet with the greatest consideration for the volition of his fellow-men.

So in these considerations a fundamental fact is expressed, namely, that there does not exist a single class in the mighty diversity of our experiences and activities that could not be subjected to scientific examination,—in which, in other words, one could not work out the recurrent regularities and use them for the prediction and, where one may exercise any influence, for the pre-formation of the future. So on this side, too, science shows its specifically human and social character in a way that would be impossible for either applied or theo-

retical science of any importance so long as the human individual has to depend upon the narrow compass of his own powers and upon the short duration of his personal life. Only by means of the process of socialization, by means of the possibility of communicating one's own experiences and the generalizations derived therefrom to posterity, and indeed, by means of writing, to communicate them for any desired length of time to posterity, independently of any personal factor, has the enormous development of science become possible, of which we are the surprised witnesses as we contemplate the history of recent and more recent times.

These observations, finally, define our position as regards history. Owing to the circumstance that the civilization of central Europe has been erected upon the half-lost traditions of ancient Greco-Roman civilization, the means for attaining a knowledge of that old civilization, which appears so inaccessibly lofty to those striving after it, have enjoyed quite special prominence. And since from the nature of the case it was only a question of phenomena of the past, the means for investigating the conditions of the past and for bringing them to the knowledge of the present age came into correspondingly high repute and have undergone very extensive development. This explains the great respect which all historical disciplines have enjoyed. To begin with, historical disciplines which had to do with scientific, artistic, and religious traditions were, as a matter of course, appreciated to an extraordinary degree. Then this valuation was extended involuntarily and automatically to the investigation of all possible forms of culture of a higher and of a lower degree which were being rendered accessible by means of the same instruments of historical investigation. As almost always happens in human affairs, the means finally became confused with the end, and became in themselves the

object of endeavor, in such a way that the present intellectual tendency of a great number of scientific persons has led them to the point of looking upon a merely exact knowledge of the past alone as an important task of science and worthy of any sacrifice. In reply to this it must be said that historical investigation in itself cannot be considered by any means as a science in its own right. History must rather be looked upon as a scientific technique, as an auxiliary means for the development of science, which, in an especial way, finds application to every individual field of all science. What is now called history was until recently almost exclusively history of rulers, states, and wars, and had reference, therefore, to an exceedingly insignificant part of actual events. Slowly and with considerable resistance on the part of those concerned, the idea has been making headway that the history of technical science and of civilization is a far more important discipline than the history of wars and countries. But as a natural result, again, of accidental historical development, the history of civilization is understood to be rather a history of art, of belles-lettres, and the history of the disciplines connected with them as a history of techniques; whereas every unprejudiced survey of the development of peoples and states teaches us that this development is preeminently determined by the technical agencies and capabilities at the disposal of peoples and states, while the artistic-literary side has played relatively only a secondary rôle therein.

Hence a logical history of civilization would be above everything else a history of technical science, and the history of the other intellectual possessions, of religious ideas, of art, and of science would have to be incorporated only as special headings in this general history of human progress written from a technical point of view.

Accordingly we see that an investigation of history would presuppose a still more varied preparation than that demanded above for the philosopher of the future; that is to say, in addition to a wide and fundamental knowledge of all the theoretical or general sciences, it would presuppose a much more detailed knowledge of all the *applied* disciplines, from astronomy to chemical technology and to the theory of natural selection. It is evident that the only attitude mankind in its present stage of culture can take toward these questions is *that the technical science of historical investigation is connected as a scientific method with the pursuit of every single discipline.* And heretofore, moreover, things have so shaped themselves in many places involuntarily. For example, we have historians of mathematics, namely, mathematicians who by means of historical investigation and with philological knowledge and the methods of literary criticism have thrown light upon the history of this particular discipline. In the same way the history of chemistry down to the present time has been written exclusively by chemists and not by specialists in history, for the simple reason that the professional historians have not the necessary knowledge.

What has been brought about automatically in this matter under pressure of actual conditions should now be cultivated farther in a conscious and scientific manner. In each individual discipline, in every pure science as well as in every applied science, the historical part should be submitted to careful scientific study. But it must be particularly noted that this should be done only from the universal point of view of scientific work in general, namely, for the purpose of utilizing logically and methodically the knowledge of the past for discovering general laws and at the same time for predicting the future. The definition given by the cele-

brated German historian, Leopold Ranke, which exercised upon a whole generation of historians an exceedingly narrowing and enervating influence—namely, that the only important thing for the historian to know is how things have come to pass—must be rejected for fundamental reasons. We have not the slightest interest, in and for itself, in knowing what has occurred in the past, for we have not the least influence on this past, and even the most accurate knowledge of it does not enable us to change it in any way desired by us. Only *in so far as the past* has *future value*—that is, only in so far as one is able from a knowledge of the past to deduce universal laws for shaping in general the field in question, and can apply them for predicting and, wherever possible, shaping the future in the general interest of mankind—have historical studies meaning or a right to existence. If one surveys the present pursuit of many disciplines from this point of view, one will become convinced that even in the twentieth century we still suffer in various ways from unproductive scholasticism, from pseudo-science, which has arisen everywhere from the fact that the means have been confounded with the end, and the correct bearings have been lost as to what is and what is not worth knowing. The past is infinitely too rich in events ever to be exhaustively reproduced even by the most careful and most complete study. For, at the very time we are devoting all our intellectual powers to such study, there actually happens in a moment so enormously much that to try to reconstruct in all its details any part of the past seems like drawing water into the vessel of the Danaides,—the mighty sea of new occurrences at once covers up all islands of this kind, islands that have been won with difficulty. So the essential impossibility of such a task in itself demonstrates its essential impracticability. On the other hand, the question of what relationships, what uni-

formities, what general formations of concepts can be deduced from the knowledge of any past events whatsoever affords us a safe guide that teaches us to judge what fields in the past and what problems of historical investigation are really worthy of study, because there finally results, not the science of the past, but the only science that deserves the name—*the science of the future.*

Second Lecture

PRINCIPLES OF THE THEORY OF EDUCATION

LET us attempt to picture to ourselves how pedagogy, scientifically systematized, will look in the future. To think of thus anticipating the future is not as presumptuous as might at first appear. For, by means of a methodology that has recently been developed, common to all the sciences, we may also examine the classification and content even of those growing sciences which, on account of extraneous influences, we have not yet been able to develop to the extent that the general scientific and cultural conditions of the age would warrant. We shall first have to occupy ourselves with calling to mind in a short review the entire system of the sciences. With the help of this system we can then answer the question where pedagogy is to be classified. Then, by reason of the place which will be assigned to pedagogy in the system of the sciences, the systematic arrangement of this discipline may be readily deduced according to established principles.

I would call attention to the fact that the totality of the pure sciences may be divided into three groups—the sciences of *order,* the *energetical* sciences, and the *biological* sciences. The sciences of order begin with logic, or the theory of classes; they include, moreover, mathematics and geometry as well as the science of time, which has not yet received a distinctive name. The energetical sciences include mechanics, physics, and chemistry, and, as is well known, have as their chief characteristic the idea of energy, which as yet plays no rôle among the sciences of order,

having made its appearance as a new subject of study in this second department of science. The biological sciences, finally, are to be divided into physiology, psychology, and "culturology" (*Kulturologie*), the first having to do with the most general phenomena of life, the second with those special phenomena called processes of the spirit or mind, and the third finally with the biological-psychological phenomena which occur exclusively or wholly in the highest species of living beings, man. These specifically human peculiarities which differentiate the race of the *homo sapiens* from all other species of animals is comprehended in the name *culture;* therefore the science of specifically human activities may be most suitably called *culturology*. It coincides practically with what has been called sociology. This name, however, is not entirely appropriate. The fact of association, to be sure, is extremely important for the development of human culture; but, on the one hand, it is not the only determining factor in this field, and, on the other hand, there are so many kinds of associations among animals and plants, and even among minerals, that one cannot employ the idea of social organization as a specific characteristic of this highest of the sciences.

Now, there exists between the sciences just mentioned the relationship that the first mentioned more general sciences always have an influence and sphere of application in all the sciences that we have mentioned: physics finds its application in chemistry as well as in all the biological sciences, but no application in mathematics, logic, geometry, etc. The higher, therefore, a science stands in this succession, or the later it has been named, so much the more do earlier, fundamental sciences come into consideration in connection with it and contribute to its content for classification and examination. While, for example, chemistry employs as aux-

iliaries or presupposed sciences only logic, mathematics (including geometry), and physics, there come into question in every individual science belonging to culturology, one after another, *all* the sciences, and from this there result naturally a division and an exhaustive view of all the problems which are to be solved in that particular science.

Before we pose the chief question and its answer, to which of these domains pedagogy belongs, we must state by way of premise that, by the side of the pure sciences just mentioned, there are a great many special disciplines that are called *applied* sciences or techniques. They share with the chief sciences the application of the laws of nature, but differ from them in that their goal is not systematized learning and order, but some practical problem the solution of which has forced itself upon man as a necessity. So medicine, for example, is that kind of an applied science or technique that makes abundant use of all the sciences up to and including physiology, and in some of its disciplines it employs also psychology and the science of civilization (*Kulturwissenschaft*). Each applied science has, like medicine, its fixed center of gravity in one of the pure sciences. It will therefore, as a matter of course, use in its functions all the more general or subsidiary sciences also, while little—under certain circumstances nothing—from the higher sciences comes into consideration in connection with it.

We have now made the necessary preparation to enable us to designate exactly the position of pedagogy in the whole system of the sciences. In the first place, there is no doubt that pedagogy belongs to *culturology*. As we have already seen, pedagogy is concerned with handing down the culture of the present living generations to the ones coming next. We recognize, furthermore, that pedagogy is an *applied* science, since it is not a question here of purely perceiving,

systematizing, and ordering any natural facts, but it has rather as its purpose the influence to be exerted on the growing youth in the manner often described. *Pedagogy is, therefore, a chapter of applied culturology or sociology,* the pure sciences appearing as sciences subsidiary to it, since culturology as the supreme or ultimate science is in its way dependent upon all the earlier or more general sciences. We shall, therefore, get a view of the whole content of scientific pedagogy if, with respect to pedagogical problems, we inquire into the influence and importance of each particular science.

Before investigating the relationship of pedagogy to the several single sciences that become step by step more complicated, we have still one general point to settle, the more important features of which I must at least touch upon in order in some measure to answer at the very outset questions that may possibly arise. We shall have to call to mind that the problem to which we are now turning has two different sides. I mean that the question involves not only the influence which the various sciences exercise upon the *subject-matter* of instruction, but also the influence that they exercise upon the *process* of instruction. Accordingly, both matter and form of instruction are influenced simultaneously by the various sciences, and it seems logically and methodically imperative to keep these two sides always distinct. I should anticipate right here, and say that such a division does not influence the results of our examination to any great degree. I must admit that I myself was astonished to see how much these two questions merge and hang together, thanks to the general method to which they are here subjected. From this fact we may draw the general conclusion that on arriving at a really rational solution of the problem, the two phases of pedagogical science will show

themselves to be closely related; that, in other words, the content of instruction determines its method adequately and absolutely.

We recognize this most clearly, perhaps, in the very first heading to be treated, in the relation between logic—or rather the theory of multiplicity—and pedagogy. Owing to a strong movement which is making itself very evident in present-day science, the former unqualified veneration for the Aristotelian logic has been giving place to the more recent notion that what we with justice call logic is nothing more than the theory of the most general and most commonly recurring relations among different things and their concepts. Accordingly, there come into question, not only the manner and means by which from two propositions a third may be construed (the exclusive content of logic hitherto), but also a more universal problem. What modern logic treats of is how things may be classified, how the resulting groups may be mutually co-ordinated, and what results and laws ensue therefrom. From this point of view we see at once that the whole province of human speech belongs in this large general chapter. Speech is nothing but a system of signs which we associate with the system of concepts, and which we have formed for ourselves for the purpose of transmitting our ideas to others by means of language. Language serves, therefore, for the *communication of ideas* by means of the following process: Some definite sign is associated with a given idea, and this sign must always be the same for that particular idea. If, now, another person is led to connect the same ideas with these signs, he "understands" the language in question; that is, on recognizing the signs, he forms in his mind the same idea that the first person had in mind when he produced the sign. We are thus concerned here with an unusually general and therefore im-

portant case of the association of two groups, the concept group and the lingual sign group.

Soon, however, this phenomenon assumes another aspect by reason of the fact that we are not able to satisfy ourselves in all cases with spoken, phonetic symbols, lasting but a moment; we find ourselves, on the other hand, forced for many purposes to associate enduring symbols with our ideas, such as are employed in our written language, *script.*

This means, pedagogically speaking, that what the child must first acquire is the ability to form and employ a number of ideas that are connected and sufficiently clearly grasped, as well as the ability to associate with these ideas the conventional speech symbols of its mother tongue. Only after the co-ordination of words has been completely established and fluently learned can the association of written symbols follow. Along with this a new fact makes itself felt, namely, that when a group *A* is associated with a group *B,* and again a group *C* with group *B,* then groups *A* and *C* prove to be co-ordinated. Now, if one represents the symbols of speech by means of letters, without concerning one's self about the sense of these sounds,—that is to say, without reference to their associated ideas,—one obtains again a system of symbols, the written language, which is co-ordinated quite as closely with the original ideas as the sound language was.

In methodical presentation all these things look rather abstract and uninteresting, but they assume at once a concrete form as soon as one envisages the real pedagogical problem in connection with the child-mind in process of development, viz., on the one hand, the formation of clear and precise—that is to say, sharply differentiated—ideas, and, on the other hand, the association of the symbols or of the words with these well-understood concepts. Obviously,

this analysis gives as a result (and in a way which is to most persons as unexpected as it is illuminating) the principle of the industrial school (*Arbeitsschule*). The phrase "industrial school" is one of those phrases the co-ordination of the meaning of which is not quite clear, probably because there is such a lack of definiteness to overcome in the idea itself. In the light of the observations just made, one sees that the industrial school comes to mean in the lower grades that the cultivation of the *formation of ideas* within the range accessible to a child proves to be the first and most important mission of the school. Therefore, during the first year, when the teacher with his class looks about him chiefly in the schoolroom, afterward in the house, street, or field, and describes the various objects and situations and gives names to them, he is carrying out the most elementary pedagogical application of the first and most general of all the sciences—the science of co-ordination, or logic. At the same time we recognize the fact that the method hitherto employed—*i.e.*, that of beginning to teach reading and writing as soon as possible —is shown, in the light of this analysis of earliest school activities, to be unfit for the purpose. The most important thing, because fundamental, is first the formation, co-ordination, and clarification of *ideas;* and since this work necessarily forms a basis for all else, a corresponding amount of pains and care must be bestowed upon it. Above all, it must not be demanded of children that they represent and reproduce ideas before they have grasped clearly the content of the idea itself. Only after the child, within the range of its experience, can express itself fluently about the ideas that present themselves and about their mutual relations,—when, for example, it can relate its little experiences coherently,—then only does the question present itself how we are to coordinate with these abstract representations expressed by

sounds those expressed in lasting symbols. In other words, one will not begin with instruction in writing and reading till about the end of the first year, and this year will be devoted almost exclusively to the cultivation of the processes of conception.

Now as to the association of the written and printed symbols with the ideas and their sound-symbols, we know that German schools suffer most seriously by reason of the variety of alphabets, the number of which (large and small, written and printed, Roman and Gothic) amounts to no less than eight. I do not wish to permit this opportunity to pass without again urgently calling attention to the fact that, in the first place, the so-called German, or Gothic, script has nothing to do with the idea of "being German," and, in the second place, it proves to be a serious hindrance to the mental development of our children and of our people. Not even a half-educated person may avoid learning the Roman script, printed as well as written, because, for example, all the notices on the railroads, the street corners, etc., and the characters on all typewriters, are in this script, for the very simple reason that it is much easier to read than the Gothic script. The knowledge of Gothic script is therefore necessary only for written and printed statements in which it is still injudiciously retained, and in which its discontinuance is only a question of time. While it is true, for example, that in our daily press most of the general news items are still to be found printed in Gothic characters, nevertheless news of scientific and commercial character, which it is presumed will be read by readers of other nationalities, is already being printed quite generally in Roman characters. It would not occasion the slightest difficulty, but rather bring about far-reaching relief in the ratio of 2 : 1 in the instruction of children, if one should forego completely all knowledge of the

Gothic script in the early years of school, and leave acquiring a reading knowledge of it (for writing it is completely superfluous) till a riper age.

This explanation of the material content of primary education gives us at the same time an insight into the corresponding pedagogical procedure. At this stage the teacher will, above all, see to it that he promotes to the best of his ability the formation of concepts, that he compels the children to represent accurately to themselves the characteristic and constituent parts of the various ideas they possess, and he will also take care that the very clearest and most definitely co-ordinated words are employed for the clear and definite ideas thus obtained. Just here many difficulties may be encountered, for all the so-called natural languages (*i.e.*, those which have developed unsystematically) leave much to be desired as regards order and regularity, and therefore often violate the prime requisites of logical association—above all, the necessity of *avoiding ambiguity*. The capacity of the teacher will be shown by his skill in overcoming these internal difficulties of our present language, and in pointing out to the children the existing obstacles and ambiguities in order that these may be avoided. As an example of the extent to which the material analysis of the content of instruction also elucidates the method of imparting it, I need only mention, in passing, that in the logical analysis of first conceptions it is necessary, from the point of view of method, to separate learning how to write from learning how to read, and to place the former, as the more difficult art, at a later period, when the relation between the idea and its written symbol has become completely comprehensible.

After this first division of the sciences there follows quite naturally the second—the theory of quantities, or mathematics. Reading, writing, and arithmetic are the traditional

subjects of elementary instruction, and in the very same way experience has brought us to realize that arithmetic should not come until considerably later, when the more general elements in the systematization of mental processes have been treated and made familiar to the child. One cannot build up the science of mathematics logically, nor can that science, therefore, be rightly understood if one has not first acquired a clear grasp of the ideas appearing in the sphere of order, and of the mutual relations of those ideas.

It is also a matter of common knowledge, and one that is making more and more headway in our day, that after arithmetic should come the simpler elements of geometry. Of course, for definite logical and philosophical reasons, geometry should not be taught in the form of Euclid's Exposition, but rather as an empirical science, which it certainly has always been. It is just here that the special side of the modern industrial school makes itself felt, where the child, by handling objects of dimension, by producing them from plastic material, and by their respective transformation during the process of alteration, will acquire a quantity of notions regarding practical geometry at an age when the usual instruction, by reason of statements that are abstract and (for a child's mind) too general and empty, would lead nowhere. Helmholtz himself states that the principal theorems of geometry were perfectly familiar to him the first time they were taught him in school, and that this was due to the blocks with which he had played during his early years, when he was repeatedly compelled to keep to his bed.

With these hints, let us leave for the present the subject of the sciences of order, as regards their influence upon the theory of teaching. In connection with the reflections just made, a hundred other relationships, which naturally follow

from what has been said here, will have suggested themselves to every teacher, but they cannot here be analyzed singly. For as soon as one begins to undertake a systematic examination, the subject-matter grows irresistibly into a complete system of pedagogy, the analysis of which would require not the few pages at our disposal, but volumes.

We shall now turn to the second department of our classification, the *physical* or *energetical sciences*. In case we desire for symmetry's sake to preserve the tripartite division, we may divide them into *mechanics, physics,* and *chemistry*. It is a matter of common knowledge that it is customary to impart these sciences to the child only in the higher grades of instruction, but their relatively early position in the realm of all the sciences suggests the possibility that by postponing instruction in these sciences in the case of the developing child we have not heretofore waited too long. Here, too, we must distinguish very carefully the pedagogical presentation of the subject-matter connected with the daily concept formation of a child, by which at first only the general content and general processes of the science are demonstrated by means of natural examples from the exact, logically ordered, and systematic presentation of the whole science. The sciences to be considered in this connection are already so highly developed theoretically that even their general propositions may be made perfectly intelligible to a child. I need only remind you that elementary mechanics of the spade, wheel, lever, and hammer, gives a sufficiently complete introduction to the idea of energy, and can explain quite interestingly, even to the slightly developed mind of a child, the law of the conservation of energy by means of the general principle of the conservation of work in machines. I recall, in my own development, that at the age of twelve or fourteen I was sufficiently advanced not only to incorporate

these things in my memory, but also to find that inward intellectual pleasure in their arrangement and in their respective relations the production of which is the most effective auxiliary of every good teacher.

These hints concerning the content of instruction, in so far as it has to do with the energetical sciences that I have mentioned, may suffice. On the other hand, the application of suitable fundamental concepts in the method of instruction is deserving of some attention, even though it does encroach to some extent upon the subject-matter of physiology. The teacher must accustom himself to treating the child as an energetical machine,—which, like every other organism, it really is,—and to conforming his treatment to the principles of energetics, whose first and most important axiom is that perpetual motion is an impossibility. In other words, it is not possible to produce work out of nothing; but rather, the only way to realize work consists in transforming other stores of free energy into the needed form. Unfortunately, the pedagogy that has been practised up to the present has paid very little attention to this basic law of all natural phenomena, though it circumscribes the sphere outside of which nothing can ever occur. Our present pedagogy is predominantly pedagogy of the *will*. By working upon the will of the child, with a view to reward or punishment, we have attempted to attain the desired results, and in case of failure to supplement it with all the more powerful influences upon the will in proportion as the real performances fall short of those desired. I do not mean to intimate that we should fail to recognize that influencing the will is a factor, and a most important factor, in all pedagogy. This influencing, however, is possible only *within* the scope of the laws of energy; and where the demands involve an infringement of these laws, even the most powerful influence on

the will accomplishes nothing. If a child that has not had sufficient sleep and is underfed is expected to do normal work in school, and if the teacher, either voluntarily or by virtue of the regulations, forces the child to do that normal amount by influencing its will-power, it is a question of nothing but the attainment of perpetual motion, the possibility of which, however, is excluded by the most important synthesis that science knows. A child that brings with it no store of energy into school possesses also no forms of energy which it can transform into the work demanded, and all the influences brought to bear upon the will, from affectionate admonition to severest punishment, cannot alter the situation in any way.

The beginning of a more practical understanding of these conditions is beginning to make itself felt in many directions, in so far as care is taken, thanks to charitable foundations, to provide weak and underfed children, before the beginning of school instruction, with the necessary stores of energy by the distribution of milk and bread. But these distributions are looked upon at present more in the light of charity, and it is taken for granted that one is doing something unnecessary, whereas more careful consideration teaches that the work of the teacher expended upon children of that kind, who are provided with insufficient stores of energy, is quite useless and in its way a waste of energy. Every township, therefore, that does not see to it that working children have really something to consume, that they are physiologically capable of work, spends the money used in school instruction in exactly the same way as a manufacturer would do if he attempted to construct his product with poorly made machines, dull files and knives, and similar inadequate apparatus.

We turn now to the biological side of pedagogy. The following circumstance is to be noted, which is of consider-

able importance in the problems of method which confront us. To a growing child a beetle is relatively much more interesting than a stone; the attitude of other people toward it is much more a subject of notice than possibly the phenomena of the clouds or the actions of electrically charged Sambucus balls. This is a circumstance which is connected quite naturally with the formation of concepts. The child forms its first and most familiar concepts in immediate conjunction with its daily experiences; that is to say, other people are, above all else, absolutely and indubitably objects of interest to it, and other interests follow these only in proportion as they are conceived as being less and less like man. For this reason it is much easier in school to awaken an interest in animals and plants than in minerals and physical experiments. So a certain antithesis makes itself felt between this natural organization of the human mind and the logical construction of the sciences. For the latter begin with the most abstract ideas, those lying farthest from the developing mind of the child, and ascend from them to the ever more varied and therefore more comprehensible ones lying nearer to the perception of the child. This seeming contradiction is explained by the fact that the former more general branches of knowledge, as has been explained already, are introduced as entirely empirical subjects. By no means do we dare entertain the idea, so far as a child is concerned, of an exhaustive, systematic presentation of them; but we should rather make use only of those parts thereof which, in the daily life of the child, prove themselves to be necessary and therefore familiar, and in the end also interesting. And so at this early age one will not wish to give a systematic presentation of physics and chemistry, but will, of course, rather familiarize the child with the fundamental phenomena that daily life brings with it, without special

reference to formulating them. At the point where the two divergent lines almost meet—viz., where the conceptual faculty of the child has already advanced to a knowledge of animals and plants—the contact with more general and more abstract concepts will be brought about by ascending the scale of science and by the diminution of childish interest. That zoölogy and botany can be taught with success at an age when systematic physics or chemistry could not yet be taught is due to the fact that, in presenting to children at that age the science of animals and plants, the exposition is limited to their appearance, and to those circumstances of their being which resemble similar functions in man. It is merely a question of the continuation of the theory of concept formation to which we referred in our discussion of the most elementary stages of systematic instruction. Instruction in the physiology of animals and plants cannot be accomplished otherwise than upon the basis of a sufficient knowledge of chemistry and physics, and should be postponed to a very much later period.

So much for the content of instruction as regards the departments just mentioned. The method has been touched upon already to some extent, my observations regarding the energetical side of the question having suggested the premise that a child is a living being, a biological organism. At all events, we may add here a few additional remarks growing out of the physiological and culturological phases of the subject. As far as the application of physiological laws to the method of instruction is concerned, this is a department of knowledge that has begun to be opened up only very recently. It is less than a decade since we began recognizing that all pedagogy presupposes the knowledge of psychology in its application to teacher and to child; that all scientific pedagogy, therefore, must begin with the study of

child psychology and the psychology of the processes of instruction. This, of course, is not the most important influence that psychological knowledge has had upon pedagogy. On the contrary, the highly endowed empiricists of the past, to whom we are indebted for the best in all that has been accomplished and systematized heretofore, recognized these fundamental relationships long ago as a matter of course and put them to practical use; as, for example, the precepts that the simple must precede the complex, that the mind must not be fatigued by being occupied too long with one subject, etc.

Among the empirical results thus obtained I wish to lay particular stress upon one only, because to my mind too little emphasis has been laid upon it, although it is absolutely fundamental for the successful solution of the problem of teaching. In the course of an investigation undertaken years ago for an entirely different purpose, I attempted to account for the principles which, in conformity with natural laws, might be established as regards the most general problem of every human life, namely, the attainment of *happiness;* and I came to the conclusion that the most important requisites for happiness are, first, the greatest possible amount of completely transformable energy, and, secondly, the greatest possible amount of energy transformed voluntarily.[1] The workings of human energy may be divided into two parts: one that is transformed in complete conformity to the actual will of the person in question, and another that is brought into transformation under the influence of compulsion of some kind. A life filled only with forced activities repugnant to the will is felt by everyone to be a condition of the greatest unhappiness. On the other hand, the various prov-

[1] Cf. "Die Forderung des Tages" (Leipzig, Akad. Verlagsges., 1911), S. 217.

erbs on the subject of happiness reveal the fact that activities which are in conformity to volition have long since been recognized as the absolutely necessary premise to every sensation of happiness. But at the same time, energetics also teaches that the result of every transformation of energy depends, first, upon the total amount of available energy, and, secondly, upon the quality-ratio—*i.e.*, upon the proportion of raw energy that can be transformed into the form desired for the particular purpose. Accordingly there appears a remarkable parallel between the quality-ratio and the sensation of happiness; that is, the highest quality-ratio is attained when the transformation ensues with the least resistance: for every resistance that must be overcome consumes an expenditure of energy which must be withdrawn from the principal objective. In the same way, happiness increases with the diminution of resistance. From this it follows that in school the children will accomplish a maximum of work when that which they do is accomplished with the least resistance, under the least possible coercion on the part of the teacher, and hence with a maximum of sensations of happiness. *Therefore, in the feeling of happiness on the part of the pupils we have a means of measuring the expediency of the instruction itself.* The happier a pupil feels during the recitation hour, the greater will be the success that the teacher may expect from his instruction during the period.

These analyses in the domain of psychology and energetics coincide with experience in accordance with which those teachers who understood how to train their pupils to joyous and enthusiastic participation in their tasks actually had also the very best success in their instruction. Not only is it a fact that children are accustomed to cling to such teachers with lasting gratitude, but that the immediate and

concrete results of teaching under such circumstances are incomparably greater than those obtained by severe teachers through employing coercion. If children are forced against their will to work on assigned tasks, only transitory results at best ensue. The children cram their minds with the subject-matter demanded for the quizzes and examinations; they forget these things learned unwillingly very quickly, however, and the result is nothing but a lasting detestation of the teacher and a vacuum in the ill-treated brain.

In order to make this important principle clear by an example, I should like to recount a purely empirical and unintentional confirmation of it, for which I am indebted to the well-known pedagogical reformer, Berthold Otto. Berthold Otto describes what he calls *Gesammtunterricht* ("joint instruction"), a system discovered and developed very completely by him, by which the children themselves, with their questions and observations, assume the conduct of the exercises of instruction and the teacher is present merely to maintain a sort of parliamentary order (which requires but very little oversight) and to give the actual information which the children do not possess, and for this purpose he employs as an aid either his memory or an encyclopedia ready at hand. Students of pedagogy who visited his classes afterward complained to the leader that the children sat around in such disorderly fashion, that each particular child sat at its desk in a different attitude, and nothing at all was to be seen of the order that is carried out in a military way in a normal class. Otto was accustomed to answer this by saying that in the beginning he, too, had endeavored to bring about greater uniformity; but the difficulty of attaining orderliness while the children were following with eager interest the content of the subject-matter under discussion had led him, from a pedagogical point of view, to forego this re-

quirement. And not until I had called his attention to the fact that the matter was a simple question of the law of the conservation of energy, that a child could not at one and the same time give its complete attention to the content of the questions posed and give heed to the position in which it sat at its desk, and that a demand made in the one direction necessarily resulted in a diminution in the other,—not until then, I say, was a theoretical motive shown for that which his pedagogical instinct had led him to see was right.

The same observations may be made concerning the much discussed question of the independence of the teacher as regards the managing of the children and the treatment of the subject taught. If one watches a group of workmen at work, one will find that almost every one of them handles his tools in an individual manner. This is due to the fact that all men differ from one another, and that the conditions under which each of them uses a particular tool most practically must accordingly differ from one another. The dissimilarity of teachers in mental as well as physical organization must necessarily cause their methods of instruction to differ. Every form of coercion that does not take these personal differences into consideration, and that seeks to bring about a uniformity not justified by weighty reasons, only serves to diminish in the teacher the quality-ratio of the work of instruction. From these observations we shall have to conclude again that uniformity is to be striven for only in so far as it is shown to be urgently necessary for the organization of the school system as a whole; that in drawing the line between freedom and constraint, however, it is better in case of doubt to place our line of demarcation more toward the side of freedom than toward the side of uniformity. In such circumstances there exists a greater probability of better quality-ratio in the functions of the system of instruction as

a whole, and this, after all, should be the chief aim of all educational administration.

Yet another great division of our subject remains to be treated finally, namely, that of the application of *sociology* to the school. For even though pedagogy comes under the head of applied sociology, this does not mean, after all, that it stands isolated from the other branches of this science. On the contrary, since it is a question of applied method, we must investigate the entire range of sociology in connection with its influence on pedagogy and its method of application. We see at once that we are confronted by an almost inexhaustible problem. Here again we shall have to content ourselves with a few brief suggestions as to how far the application of scientific sociology influences, on the one hand, the method, and, on the other hand, the content, of instruction.

Now as to method, it is a question of consciously linking our growing youth, by means of education, to the whole cultural fabric of the present; and one sees at once how far-reaching and elucidating is the light which falls, owing to this relationship, upon our present educational system. Quite an important part of secondary education (a short time ago one had to say the vastly greater part of it) hinges upon the acquisition of the two ancient languages, Latin and Greek; and the so-called humanistic or rather philological gymnasium insists upon the tenet that by the acquisition of these languages, and of the old culture of the Greeks and Romans connected therewith, by far the best means for the attainment of culture is placed at the disposal of modern man. The former point of view, that the culture of the ancients was so incomparably superior to all other possible cultures that we for our part can only hope to attain to a certain degree of perfection by the imitation of their attainments, is still prac-

tically held; theoretically, however, it has essentially been given up. For the representatives of the philological gymnasium are now attempting to establish their system upon essentially different grounds, since both classical philology and archæology have also begun to incorporate the history and works of the Greeks and Romans in all the events of the history of the world, and especially to occupy a more critical attitude toward the products of their art and philosophy. On general principles, the following must be emphasized in this connection: The fact that man is a being capable of development, that his present conditions of existence are therefore better, nobler, more favorable—in a word, more valuable—than his earlier circumstances were (all of this to be taken on a basis of general average), necessitates a different appreciation of old things as compared with the new. Owing to a natural error whose obviously possible origin I have explained in another connection,[1] we have arrived at an overestimation of old things, in comparison with those that we now have, which has repeatedly led to confusion of thought in our cultural work and prevented us from attaining a right point of view. If we look at the matter simply and soberly, it does not admit of a doubt that those peoples whom we are accustomed to call the ancients were actually young; they lived some thousands of years earlier than we, and our civilization has been able to evolve by employing all the cultural results won by those earlier and younger peoples. Expressed in other words, this means that, viewed from the stage that we occupy in man's development, we *people of the present day* are the *oldest* of all peoples, the ripest, the most developed, the people who culturally stand highest, and all other, earlier stages of human development stand, as compared with the present, in a backward position

[1] Cf. "Die Forderung des Tages," S. 282.

for the reason that humanity at present can and may use what humanity in the past has laboriously produced. In this connection, of course, we must take into consideration the fact that the road upon which man has developed in the past, especially in the earlier ages, was not a road continuously rising, but rather it ascended and descended in great waves. There arose, especially after the destruction of the culture of antiquity by the migrations of the nations, a cultural vacuum for the filling of which the remains of earlier culture first had to be drawn upon. But in the meantime we have long since passed beyond this void. Great new and fundamental realms of culture, especially in the sciences, have been disclosed; and should we compare our present condition with that of the Greeks, or even of the Romans, we could boast without any exaggeration of a much higher degree of progress. The single circumstance alone, that for hundreds of years mankind has been taking into its service various kinds of inorganic energy, especially from fossil coal, means such an enormous freeing of human labor from the monotonous toil of muscle without any addition of mind, that by this very fact alone our claim is established that we stand upon quite a different height of culture from any that could ever have been reached by the peoples of antiquity. Did not Aristotle, in complete accord with the point of view of his time, emphasize the fact that slavery would never cease because one could not otherwise conceive how the rest of humanity would be able to get flour for their food? So, in the last analysis, one sees there is nothing in making use of the civilization of antiquity as our highest ideal of culture. An ideal, as I have often explained, can lie only in the future, never in the past; and every ideal that is artificially sought in the past is only a means of reaction, and is from its very nature inimical to culture. Thus we are experiencing in our own day the fact

that the philological gymnasium is irresistibly approaching gradual extinction, in spite of the constant and inconsiderate support given to it by the ruling reactionary classes in Germany, for obvious political reasons. The contrast between the cultural needs of our time and the cultural means that the philological gymnasium can transmit is too great and glaring for this remnant from the Middle Ages, that was skilfully galvanized anew a hundred years ago, to be kept alive in the long run. The yearly increasing attendance at the non-philological institutions, as is shown by statistics that are more and more favorable to them, speaks in a language which in this connection is not ambiguous.

Now, by an illogical application of the fundamental biogenetic law, they have attempted to justify the education of our growing youth through the example of the Greeks and Romans, by saying that just as every organism must by way of short review pass through the various stages of development of its species, in the same way it is also necessary to mental development that our children who are destined for higher education should also pass in school through the earlier stages in the development of humanity. If this were true, and if this argument were taken seriously, then our poor young gymnasium students would first have to learn Babylonian and Egyptian culture and history before they could be introduced to the joys of Latin and Greek grammar. None of the educationalists have dared this consistent application of the argument which they have employed in the defense of teaching Latin—*i.e.*, not one of them has taken his own argument really seriously.

We must repeat, therefore: *all the lower and higher school instruction at present must be determined absolutely by the cultural needs of the present time.* The lack of sociological training which has been brought about by our con-

fining ourselves to the culture of antiquity, and hitherto to a purely external presentation of history that is principally related to wars and battles, and to the establishment and fall of empires, cries for immediate remedy. It has arisen from a completely false conception regarding the factors of culture, and the history of governments must be supplanted by the *history of civilization.* A modern child would learn what is infinitely more valuable and useful for its future life if it acquired an accurate grasp of the development of agriculture, of mining, of transportation, of the steam-engine, and so forth, than if it learned by heart all the battles of Julius Cæsar down to the last details.

The fact that this real history of man's development—the history of the conquest of nature by mind, or the history of technology and science—is as yet hardly written, to say nothing of its being taught in the schools, is clear evidence of the small extent to which the fundamental sociological facts have been employed heretofore as a subject of study in education. The modern call for instruction in sociology merely reflects the fact that we are becoming conscious, little by little, of this oversight, and are now seeking means (not always the most apt and suitable) to fill out this baneful gap in the training of the modern pupil. In fact, it is almost unbelievable when one calls to mind the present situation in all its naked truth. The very pupils who are destined in one way or another to be hereafter the leaders of the nation as teachers, judges, physicians, or ministers, do not receive during the most important period of their development in the gymnasium the slightest competent enlightenment about the ways in which the cultural, economic, and political organization of the German Empire is formed, nor how hereafter they will have to co-ordinate their civic life with its duties and its rights in the life of the nation as a whole.

And when, in conclusion, I come to speak of the influence of social science upon the method of instruction, I may encounter a reproach which often enough has been cast upon me unjustly by those who feel themselves disturbed in their present prerogatives and positions of comfort. The reproach intimates that I only know how to bring destructive criticism and fruitless fault-finding into the school question, and that I exhibit no positive or helpful activity in this field. The fact alone that until recent years I had passed my whole life—and successfully, for that matter—in educational work should be sufficient to nip objections of this kind in the very bud. But when I have characterized that which we now have as being largely in need of improvement, I have done so over and over again by giving always an exact explanation of the reasons why I considered it bad, and in so doing I have specified the exact direction which improvement must take. Just here, in connection with the question regarding the application of sociology to the technique of school-teaching, opportunity is afforded to advance a good step further in the matter before us.

Our present school organization—and this fact must be placed before everything else—is not arranged essentially with reference to the greatest possible advancement of the child, but with reference to the most convenient administration possible in the hands of the officials in charge. The thought underlying our whole present school organization is the supposition that all children are identical in character. They are received into the school at an absolutely prescribed age, and then the object to be attained is that from year to year all, in like periods of time, take up and master the absolutely prescribed portions of knowledge assigned to them, which are alike for all, so that they may advance each year to a new class, and, if all goes well, may be dismissed

after a normal lapse of time with their diplomas. Such a scheme, of course, is outwardly more readily handled than any other; for, to begin with, it is as trivially arranged as could possibly be imagined. Viewed, however, from the standpoint of its pedagogical effectiveness, it is the crudest imaginable, and therefore the most barren of result—yes, the most harmful. The supposition that all children are organized alike, that they develop with the same rapidity, that they have the same degree of interest in the various subjects, and therefore can be carried over similar distances in all the various subjects in a fixed average time, contradicts the facts in every particular. To force a system that rests upon this hypothesis can lead at best to most serious conflict with reality. For example, we are already quite accustomed toward Easter-time to hear of a number of cases of suicide among pupils, and of many other pupils who have left home secretly, either for fear of punishment or on account of shame due to a poor report.

The contention as to whether the school or the home is to blame for the situation—the reproach brought by the school, for example, that, owing to incorrect management at home, children are made nervous and irritable, and are therefore no longer able to satisfy the necessarily strict demands of the school—is fruitless. We are confronted by the situation that the present school system leads to these fearful results that are becoming worse each year, and the only conclusion that can be drawn from it is that causes which have such deplorable results must be eliminated. These causes, however, lie in the contradiction between the school organization and the actual characteristics of the pupil, between the schematizing of personal development by our school organization and the infinite variety of actual life, which is sharply opposed everywhere to the scheme. In

addition to this, there is added a frightful amount of uncharitableness toward the pupil, for which the teachers of the higher schools more than those of the elementary schools must be blamed, and this is due to the unsuitable training of the teachers. Quite a considerable portion of our teachers in the gymnasiums and similar institutions consider themselves to be Pegasuses in harness. In the university, owing to hunger for scientific activity on the part of their professors, they are led into scientific work; they consider this their real calling in life, and therefore feel that a violence is done their minds because they are forced to teach children. The result of this is that the dissatisfaction that they feel toward their profession and their work is discharged on the defenseless victims in their classes. And even though a conscious reaction of this inward discontent upon the children may take place only in very rare cases, nevertheless the unconscious discharge of this continuous feeling is a phenomenon so natural that it must be expected in all cases where it is not counteracted by an especially energetic sense of pedagogic duty.

The actual situation is a contradiction of the premises upon which our present school organization rests, the complete falsity of which its representatives have probably scarcely ever realized. Everyone who has seen a few children develop side by side realizes at what different ages the same stage of mental development may be noted. One child learns to speak in the first year, another in the third, and the same differences may be recognized in all forms of mental activity. When, therefore, a group of six-year-old children in whom a difference of almost a year may exist are, under the formal administration of regulations determined by the accidental date of birth, brought together before the teacher, the group by no means represents a uni-

form, but rather an exceedingly heterogeneous mass, and every teacher knows how varied and individual the attitudes of mind are that children assume toward instruction. As time passes, these differences not only continue to exist, but generally become more marked. The tasks of the classroom, however, demand uniformly rapid progress on the part of each individual. In order to enhance this pedagogical nonsense to the point of incredibility, a child is kept back in the class a whole year, according to the regulations, if it exhibits a deficiency in one or two subjects, regardless of how good its attainments in the other subjects may have been. This only brings about a further disarrangement of mental poise, for non-advancement means a further mistreatment of the developing child, as it is prevented from prosecuting the very things which it can perform best, and in which, therefore, it has abundantly fulfilled the demands of the school. Instead, the child must work at the disagreeable thing, and in the field to which its ardent desires are drawn it again has set before it only what is so well known that it is wearied of it. This outward schematizing in year-courses and classes, in hour-divisions alike for all, and into lessons in connection with which not the slightest consideration is paid to personality, causes the many failures in school and necessitates also the dismal phenomenon just mentioned—the complete decay of the vitality of the child owing to the circumstances under which it must live.

The question as to how one can interpose in the matter with a view to improving conditions is answered in a practical way by experience in other fields where school bureaucracy, fortunately, has not yet been permitted to make its entrance. From my own experience of almost a quarter of a century as a laboratory teacher of chemistry, I can say that by *a free arrangement of instruction as regards time and*

content vastly much more is accomplished than by the usual schematizing. If children in the schools were only treated just as we university professors are accustomed to treat our students in the laboratory, and as very young children are treated in the kindergarten, incomparably better results would be attained. Each child is set at its task and attempts to do its best with it, in proportion to its attainments and to the rapidity of its mental reactions. Just as in the laboratory we do not force the slower worker and do not hold back those who work fast, in exactly the same way children should be permitted to determine the rapidity of their development. From the general energetical reasons explained above it seems obvious that in this way by far the best results will be attained.

Only by this kind of instruction is it possible also to develop *social* acting and thinking in children. It is considered at present one of the worst school offenses for one child to help another solve its task. The one receiving the assistance is punished as well as the one who was ready to impart the help. *Is, then, mutual willingness to render help a characteristic so exceedingly general that it must be systematically done away with in school?* Is not, rather, egoism and narrow-mindedness a fault under which we suffer severely? I do not hesitate to express the conviction that a considerable amount of this illiberality is imparted to our growing youth in school by the prevalent notions regarding this mutual help and the usual treatment of it.

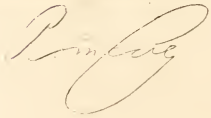
So necessary a characteristic, socially speaking, as the willingness to be of mutual assistance should rather be cultivated in every way possible by the schools. This, to be sure, is not possible in the thoughtless schematization of the present school curriculum; it becomes, however, an important pedagogic factor as soon as the system of unhampered in-

struction just described is introduced. Then those children who learn more quickly and grasp an idea more readily will become, spontaneously, most effective assistants to the teacher. To those who are more backward they will endeavor to impart comprehension of their tasks, and they will frequently succeed in this better than the teacher himself, on account of the similarity of their mental processes. In this way there develops at an early age the distinction between natures born to lead and those requiring leadership. The former are spurred on to renewed zeal in their endeavors, and in proportion to their ability they may participate in influencing their little comrades in a useful and fruitful manner; the others learn at an early age that in their advancement they have need of the assistance of the better endowed ones, and, what is the best thing for all of them, they learn subordination and how to work in rank and file.

I must unfortunately forego explaining here all the excellent and elevating results which would ensue from such a really social development of our school system. I am not the first to express a thought of this kind; for in this direction, too, the instinctive pedagogical talent, which fortunately still seems relatively more abundant in us Germans than in other peoples, has indicated the right course to a few pioneer spirits. The conception of the school as a social organization is to-day no longer so strange as it seemed ten or twenty years ago, though a century ago a few leading students of pedagogy had already taken the same decisive point of view. But the application of scientific system to pedagogical problems gives us for the first time, so far as I can see, the sure scientific guaranty that in this direction the right course for the development of our school system is really indicated. Those things that were demanded by these idealistic pioneers in the realm of education, by reason of their instinctive

understanding of the child-soul and of the cultural needs of their time, by the application of basic sociological laws to the school problem are scientifically proved and systematically co-ordinated. Therefore our age no longer needs to be forced to wait till the right way is discovered by towering individual spirits endowed with the sureness of the sleep-walker in the dark; but it behooves rather the conscious scientific thought of the twentieth century to recognize and to follow a course that results from an exact and pertinent consideration of the facts, as the mature fruit of a philosophic grasp of all human knowledge.

WILHELM OSTWALD.



HENRI POINCARÉ[1]

AMONG the various ways of conceiving man's affection for life there is one which perhaps Metchnikoff has not heretofore investigated, yet in this one way that desire has a majestic aspect. It is quite different from the way one usually regards the feeling of fear of death. There come moments when the mind of a scientist engenders new ideas. He sees their fruitfulness and utility, but he knows that they are still so vague that he must go through a long process of analysis to develop them before the public shall be able to understand and appreciate them at their just value. If he believes then that death may suddenly annihilate this whole world of great thoughts, and that perhaps ages may go by before another genius discovers them, we can understand that a sudden desire to live must seize him, and the joy of his work must be confounded with the fear of having to stop it forever.

We can imagine Abel's anguish at the thought of approaching death, when none about him could understand the ideas which he wished to propagate, and which he feared forever lost. We appreciate the moments that Galois must have experienced before fighting his duel, if we remember that a few hours before going on to the ground from which he should not return, he had not written a single line of his great discoveries.

Poincaré died at the most brilliant moment of his career, in full vigor. His spirit was young; original ideas were

[1] A lecture delivered at the inauguration of the Rice Institute, by Senator Vito Volterra, Professor of Mathematical Physics and Celestial Mechanics in the University of Rome. Translated from the French by Professor Griffith Conrad Evans of the Rice Institute.

vibrating in his brain. Did he realize that the world that filled his mind might crumble at any instant, like the wonderful castle of Walhalla, overrun with sudden fire? We cannot say. I hope, for the peace of his last hours, that he did not see death approaching; and yet the pages of his last memoir contain a sad presentiment.

His mind now is resting, and perhaps the present hours are his first hours of repose. In consideration of the activity which he displayed, the number of different questions which he treated, the new conceptions of science which he quickly absorbed, the number of original ideas which he disseminated, we are sure that he could not have rested a moment during his life. Poincaré was ever at the breach, a good soldier, until his death. During these last thirty years there has been no new question, connected even remotely with mathematics, which he did not subject to his deep and delicate analysis, and enrich with some discovery or fruitful point of view.

I believe that no scientist so much as he lived in constant and intimate relation with the scientific world that surrounded him. He received ideas and gave them by a process of exchange both rapid and intense, which ceased only on the day when his heart ceased to beat. That is why, if we were to characterize the recent period of the history of mathematics by a single name, we should all give that of Poincaré, for he has been without doubt the most widely known and celebrated mathematician of recent years. Little by little he created a type of scientist and philosopher. Without being aware of it, the mathematicians of his time, by means of subtle sympathies and bonds, grew necessarily into that type.

Scientific development, the relations of science with life, and of the general public with the scientist have been greatly

changed in these late years. The causes are easy to understand, the effects striking. Brilliant discoveries have illumined every department of life. It is for this reason that science in general has become popular, and people expect from the mathematical and physical sciences particularly, results always new and ever more useful. It may be that people even have come to have a confidence in them which surpasses their power. The scientist who a few years ago stayed hidden in his study or in his laboratory to-day mingles with other scientists and with the public. He hears the questions which are asked from every side, and he must reply. Too much urged, he must sometimes reply before his thought is ripe.

Congresses and scientific reunions have increased in number; and popular presentations and learned lectures, where people wish to know the last word of science, follow each other without pause. There is no longer any time to wait. Modern life, eager and tumultuous, has invaded the quiet dwellings of the scientists. Some centuries ago people published great volumes; they were the synthesis of the thought of a man's whole life. But that was not sufficient for the scientific development now in progress. Scientific journals to-day ask for memoirs, in which work is published as it progresses. The proceedings of the academies, short and precise reviews, have appeared. A man reports in a few words every discovery as soon as he has made it. Time presses; one fears that the next minute the discovery may be lost. But the communications of the congresses, which no one has the leisure to revise, exceed in rapidity even the proceedings of the academies and of the scientific societies. We wish to know what has not been done. We say what we hope to find. We confide that which we shall never have courage to print. This development has created a particular state

of mind among scientists, and has changed their lives, their ways of working, and even of thinking.

There are great advantages in this modern scientific life as I have just presented it. Research has become almost collective. The energies of the investigators are summed; their discoveries follow each other rapidly; competition spurs them on. Their number increases from day to day. But how many objections we can oppose to these advantages! What refinement of detail is lost! Perhaps that patience, which for Buffon was genius itself, has vanished in the tumult of the present hour. Poincaré was a modern scientist in the full meaning of the word. There was no congress, no scientific reunion where his word was not heard. Most of the scientific journals received his memoirs and the accounts of his investigations. The universities of Europe and America have heard his conferences and lectures.

A work so absorbing, so intense, may easily overdrive to the point of danger a weak or sickly constitution. Is it this excess which fatally has led Poincaré to the tomb?

Calm and serene scientific work is often a rest for the mind. The pleasure of the new results that one finds suddenly, like a beautiful landscape at the turn of a mountain road, alternates with the labor of research. The difficulties of analyzing the question are often generously compensated by the solutions which appear at the precise moment when one expects them least, by means of methods which one could not hope to find useful. The work which Euler, Lagrange, Gauss knew may be compared to a pleasure voyage in the finest of countries; but that which public lectures and conferences demand, which journals ask for at a fixed rate, very often fatigues and irritates like a long and rapid tour, during which one has no time to consider the surrounding beauties and charms.

I imagine that a mind even so largely endowed as that of Poincaré, one that possessed all the gifts appropriate to scientist and author, must have felt fatigue and weariness before a mass of labor which year after year continued without intermission or rest, and every day became more demanding and intense. But modern life called for it, and a famous man like Poincaré, most popular of mathematicians and philosophers, could not refuse.

Perhaps he felt that it was the duty of his genius towards humanity to spread abroad his ideas, not hiding any. He gave as he found, generously, as a great lord who has immense resources and is sure that no hasty expenditure can use them up. He did not hesitate between the desire to make known his thought to a great public and the fear of giving out results not yet completely ripened. An unusual lucidity saved him from mistakes. He always laid bare his ideas, and he did not hide his methods. That ingenious and subtle way of giving results and concealing the manner of getting them, so dear to the ancients and always so tempting, never appealed to him. He never waited to make complete and final his discovery, and give it a systematic and definitive form; although it is exceedingly self-satisfying to stop and investigate from every side that which one has discovered and which is really one's own. It is indeed pleasant to find new aspects of it, and obtain its applications.

But Poincaré resisted all these temptations. He sacrificed these gratifications of the scientist to a high ideal. He went ever ahead. New questions awaited him, and the time for considering the details of the old never came. Indeed, I believe that he consistently avoided details and did not wish to give his time to minute questions. It was not his business either to correct or to revise that which he had done. The whole was everything for him, the details nothing.

This inherent ardor gave to his nervous style a personal stamp and character. Perhaps it is for this reason also that it is impossible to compare Poincaré and other investigators, even those of the present date. He is too modern for any comparison to be possible. Among scientists he is like an impressionist among artists, and I know of no other scientific impressionists among the great men of the past.

It is quite certain that no theory like universal gravitation or electrodynamics will be attached to his name, as to those of Newton, Ampère and Maxwell. Among the great number of methods which he invented and developed from day to day are there any comparable with those which made famous Archimedes or Lagrange? It would take a great deal of time to distinguish everything that there is in his works, in order to say which of the seeds that he has sown will sprout, and which finally will be most fruitful. But if we ask to-day, on the morrow of his death, at what level we should place his genius, we must reply that he has reached the altitudes where dwell the great of human kind. There is certainly a philosophy that is Poincaré's, and an analysis, a mathematical physics and a mechanics that are Poincaré's, which science can never forget.

His renown during his life was great. Few scientists and a very few mathematicians have had celebrity equal to his. A physicist would find the reason for this in what I have just been saying, remarking that his spirit and the spirit of his time vibrated in unison, and that he was in phase with the universal vibration. Some great scientists have labored, urged by an internal force, without hearing or concerning themselves with those about them. They have been misunderstood. The pitch of their voices was not in harmony with that of their times, and they uttered tones which resounded only in later generations.

Nothing is harder than to prophesy the reputation of a scientist. History has given too many contradictions to obvious prophecies. Will not what one wonders at to-day be unessential to-morrow? But it is impossible that Poincaré's voice shall not be heard in future times. The questions that he treated are so important and fundamental that a great number of investigations will follow those which he commenced. His works will be studied in detail, and by many. They will form a very precious mine for all the scientists to come. The wealth of it, even at the present moment, we can surmise.

These last few words explain what I am going to talk about. It is impossible to summarize surely and adequately the entire work of Poincaré, and to give a complete survey of his mind and his wonderful activity. But I wish to devote to him this lecture. His voice should have sounded here in this solemn event, and the Rice Institute should have been inaugurated also under the auspices of his illustrious name. I shall endeavor to recall, then, a very small number of his discoveries, by trying to trace their principal characters, and to show their place in reference to the time when they were developed.

I hope to be excused if I recall facts already known, and if I consider a few details that are elementary. But since I cannot be complete I must be clear, and I shall therefore aim not to describe matters in a difficult manner. I hope that you will understand my selections from his works: I have endeavored to take them from various branches of mathematics in order to show the development of several of his speculations.

I begin with the one of Poincaré's investigations that first brought him to the attention of the mathematical world, and at once showed his great talent in analysis. This is the

theory of linear differential equations and of Fuchsian functions.

The theory of functions was the most important conquest of analysis during the last century. I did not hesitate at the Congress of Mathematicians at Paris to call the nineteenth century the century of the theory of functions, as the eighteenth might have been called that of infinitesimal calculus. An intuitive idea, like the idea of function which everybody possesses, and which is related to the most elementary conceptions of quantities which vary with constant laws, gradually has invaded the whole subject of mathematics. Analytic geometry and the infinitesimal calculus gave it a start; algebra gave a great impulse to its systematic study; and Lagrange was able to write the first theory of analytic functions, the celebrated work in which are found the germs of later progress. It is only by the enlargement of the field of variables that the theory has been built up in a precise manner. It was necessary to consider imaginary and complex values in order to be able to explain the most hidden and most important properties of functions. To study a function without considering its imaginary and complex values would in many cases be like wishing to know a book by looking at what is written on the back, without reading the pages that are inside.

Cauchy, Riemann and Weierstrass have assisted us most in the reading of this mysterious book. All of their genius was necessary to lay bare to us its most interesting secrets.

But, as often happens, a general theory can be developed only by means of a profound study of a particular class of the objects which one is considering. Always some guide is necessary to provide orientation in a new region which has not yet been explored. The guide in the theory of func-

tions has been the detailed study of elliptic functions. A great many questions of algebra, of mechanics, of geometry, and of physics lead to the development of this branch of analysis, which has followed so closely that of the trigonometric functions: the elementary functions which Euler had already shown to be related to the logarithms and exponentials.

The history of elliptic functions is well known. It has been written many times, because it is perhaps the most interesting part of the history of mathematics. We pass from surprise to surprise in passing from one step, which we believe to be the most important of its development, to another, which brings forth new discoveries and new surprises. It has happened that the general theory of functions as well as all the other particular branches which are related to it has been cast upon the model of the theory of elliptic functions, and thus it is that the theory of Fuchsian functions, which represents the latest of these constructions, follows it also, in its essential features, according to the plan of Poincaré.

As is well known, the principles upon which the theory of elliptic functions is constructed are three: the theorem of addition, the principle of inversion and that of double periodicity. Everybody has learned in the elements of trigonometry that the sine and cosine of a sum of two arcs can be calculated from the sines and cosines of the arcs themselves, by means of very simple algebraic formulæ. In its specific form the theorem of addition of elliptic functions is quite similar to that which we have spoken of. It is not, however, under this aspect that it first appeared. Fagnano, an Italian investigator who made part of no scientific circle but possessed great talent, recognized it in the geometric properties of a special curve—the lemniscate of Bernoulli.

The genius of Euler was necessary to show the true nature of this property and to develop it in all its generality.

Another most subtle property, made evident only much later, is that of double periodicity. The periodicity of trigonometric functions comes immediately from their very definition. The double periodicity of elliptic functions was not discovered until Abel and Jacobi established the principle of inversion—that is to say, when they had taken the whole theory from the reverse side. Legendre, who thought the theory already complete, had to learn that he had not yet investigated its most fundamental conceptions.

Abel and Jacobi kept on in the route which they had struck out. The general theory of the integrals of algebraic functions was systematically constructed upon the theorem of Abel, which is an extension of the theorem of addition, upon the principle of inversion which Jacobi demonstrated for the first time in complete generality, upon multiple periodicity, and finally, upon the use of certain functions which are called Jacobian functions.

The principle of inversion under a new form, the extension of the idea of periodicity, and a modified type of Jacobian function were carried over at one stroke by Poincaré into the new domain—that of linear differential equations. It was that which constituted his work of analysis upon Fuchsian functions.

After quadratures, the great problem of infinitesimal calculus is the integration of differential equations. The most simple differential equations are the linear ones. We get an equation of this sort if we imagine a relation of the first degree to hold between the displacement of a particle, its velocity, and its acceleration, the coefficients of the equation depending in an arbitrary manner upon the time. The particular equation that we have just defined is of the second

order, because the velocity is the first derivative, and the acceleration is the second derivative, of the displacement; but we can imagine linear equations where derivatives appear of any order, and which are accordingly of higher order than the second.

Lagrange and many other mathematicians studied these equations, but Gauss investigated a special class of them completely. He connected them to their series, which was the hypergeometric series. Riemann went still further into these questions. He published a celebrated paper upon the subject; and after his death results of the greatest importance were found among his manuscripts. It seems that Weierstrass, without having published anything, had also discovered much relating thereto. But we owe to Fuchs an article, appearing in 1886, which called the attention of the entire scientific world to the new manner of considering linear differential equations. If we wish to form an idea of the new level to which Fuchs and his predecessors had carried the question, we have only to compare it with the theory of elliptic functions at the time of Legendre—that is to say, before Abel and Jacobi appeared upon the scene.

And yet advances had already been made into the new subject about to be developed, since the theory of the modular function was known.

The integrals of uniform functions are reproduced with the exception of an additive constant when the variable performs a closed circuit round singular points. This property is the origin of the periodicity of elliptic functions. In the same way, the set of fundamental integrals of a linear equation with uniform coefficients is subjected to a linear transformation on going around a singular point. We seek in this remarkable fact the key to the properties of those functions which can be obtained from the linear differential equa-

tions, by a procedure analogous to that of the inversion of elliptic integrals.

If the equation is of the second order, the ratio of two fundamental integrals undergoes a linear substitution on performing a closed circuit round a singularity.

We see then that the independent variable regarded as a function of the ratio of the two integrals must remain invariant of certain linear substitutions executed upon this ratio. The property which was to replace that of periodicity was thus found, and at the same time the principle of inversion. Poincaré started from this fundamental idea and interpreted geometrically that which we have just called a linear substitution. He started a systematic study of those substitutions which belong to a single discontinuous group, because it is evident that uniform functions which remain invariant of continuous groups cannot be other than constants.

Linear substitutions correspond geometrically to transformations of the plane by means of inversions by reciprocal radii, united with reflections. They play a very important part in non-Euclidean geometry, as several geometers, among others Beltrami, had already shown. Poincaré distinguishes two kinds of groups, those which he calls the Kleinian groups, which are the most general discontinuous groups, and the Fuchsian groups. These last, interpreted geometrically, leave the real axis fixed; but by composition with a certain new substitution they leave a circle invariant. It is this circle which Poincaré calls the fundamental circle.

The finding of all these discontinuous groups is in this manner reduced to the consideration of the possible regular divisions of the plane and of space. Poincaré distinguished between Fuchsian substitutions of different families, and obtained the corresponding groups. He then had actually to construct the functions which remained invariant of the

substitutions of these groups. These are the so-called Fuchsian functions.

Jacobi, starting from elliptic functions, had arrived at a function which he called Θ—that is to say, the Jacobian function. It is not periodic, but possesses what is called periodicity of the third kind, because increasing the variable by one period reproduces the function, multiplied by certain exponentials. Jacobi showed that the simplest way to obtain the theory of elliptic functions was first to define directly this function Θ by means of a series, finding its properties by algebraic methods, and then afterwards to calculate the doubly periodic functions as ratios formed by the Θ functions.

Poincaré followed a similar method for the Fuchsian functions. He started by calculating the Fuchsian Θ functions by means of series, and then found the changes that they underwent by performing upon the variable the linear substitutions of a Fuchsian group. Certain ratios formed by these Fuchsian Θ's remain unchanged when the variable is subjected to substitutions of the same group.

It is thus that the new transcendental functions were invented. By their introduction into mathematics a new field of analysis was created. We shall not enter into the details of the properties of these new functions, upon their connection with algebraic functions, or with Abelian or other transcendental functions. Neither shall we speak of a large number of questions of arithmetic, algebra and analysis which are related to them.

But we must say a word about the relation of the Fuchsian functions with the integrals of linear differential equations that have algebraic coefficients. The direction here taken by Poincaré is similar to the one which we follow when we express Abelian integrals by means of the generalized Θ

functions of Jacobi—that is, by means of the Abelian Θ's. Following this method, Poincaré introduced the Fuchsian Zeta functions, deriving them from the Fuchsian Θ. These are transcendental functions that express the desired integrals.

It has been asked several times, Have the Fuchsian functions applications? But one can answer with the question: What does it mean for a theory to have applications? Does the touchstone of a theory consist in its use in mechanics or physics? Did the theory of conics which the Greeks raised to such a high state of perfection take its honorable place in geometry only upon the day when people believed that those curves were the orbits of planets? Was it not already a great artistic monument, without reference to any practical application?

But we must not spend time upon these matters outside of our subject. Let us now abandon analysis and pass along to other questions.

There are two kinds of mathematical physics. Through ancient habit we regard them as belonging to a single branch and generally teach them in the same courses, but their natures are quite different. In most cases the people who are greatly interested in one despise somewhat the other. The first kind consists in a difficult and subtle analysis connected with physical questions. Its scope is to solve in a complete and exact manner the problems which it presents to us. It endeavors also to demonstrate by rigorous methods statements which are fundamental from mathematical and logical points of view.

I believe I do not err when I say that many physicists look upon this mathematical flora as a collection of parasitic plants grown to the great tree of natural philosophy. But perhaps this disdain is not justified. In the evolution of

mathematical physics these researches probably are to play ever an increasing part.

Explain to a child the first propositions of Euclid. It is not the geometric properties which surprise him; rather, that it is necessary to prove them, because his mind is not experienced enough to doubt their obviousness. In the same way, certain theorems which are demonstrated in mathematical physics produce upon some people a similar surprise.

We are not familiar with the development of geometry before Euclid, and we see therefore the complete work. It is quite probable that in the progress of geometry there were periods when feelings similar to those of which we have just been speaking existed, and little by little passed away.

The other kind of mathematical physics has a less analytical character, but forms a subject inseparable from any consideration of phenomena. We could expect no progress in their study without the aid which this brings them. Could any one imagine the electromagnetic theory of light, the experiments of Hertz and wireless telegraphy, without the mathematical analysis of Maxwell, which was responsible for their birth?

Poincaré led in both kinds of mathematical physics. He was an extraordinary analyst, but had also the mind of a physicist. We shall seek for the proof of this among his works.

The memoir that appeared in 1894 in the "Rendiconti di Palermo" is one of his most interesting papers. It bears the title, "On the Equations of Mathematical Physics." The author presents the question which he is about to treat in a short introduction, where he recalls the work of some of those who preceded him. But the question has a long history of which I shall speak somewhat.

Let me begin by saying that the work has a character

which is essentially analytic, and that it belongs to the mathematical physics of the first kind. In precisely what then consists the interest of this question, which so many mathematicians have investigated? No physicist would doubt, for example, that an elastic membrane could emit an infinite number of notes, and that there would be an infinite discontinuous scale of them, going from the lowest tone to the highest. The example of sounds produced by an elastic cord or by a rod is sufficient to suggest what ought to happen when one passes from the case of a single dimension to that of two dimensions, and even what ought to result from the consideration of a vibrating body of three dimensions. But for mathematicians it was necessary to give a rigorous proof, and this proof was complicated and hard to find. We must not even suppose that the analytic investigation had the aim of calculating the pitches of the various notes. Any practical application of the calculation was quite far from the thought of the mathematician. It was only the logical point of view which gave importance to the question. Its difficulty increased its attraction and it thus became a question of compelling interest.

Physicists were intuitively aware of the result, not merely on account of the analogy of which I have just spoken, but also from a certain process of induction which has a philosophic value of the highest order, and which can be regarded as the source of several investigations which continued after Poincaré. Lagrange had devoted a chapter of his "Analytic Mechanics" to the theory of small motions. This chapter is one of the finest of his work. The author was able to carry through all the integrations in the case which he was considering, and obtained very simple and interesting formulæ. The periods of vibration of any set of molecules, finite in number, connected among each other by arbitrary restraints, were obtained by Lagrange by means of the roots of an

algebraic equation. Now any system can evidently be considered as a collection of molecules arranged in a space of one, two or three dimensions according as we consider a cord, a stretched membrane or a solid body. It is sufficient then to replace the finite number of molecules of Lagrange by these collections which we have mentioned in order to extend his results to the different cases. This is really what is called Lord Rayleigh's principle, and gives a very clear and suggestive point of view in regard to the bearing of the problem. But this principle was not sufficient demonstration for mathematicians.

The question which we have just been considering from the point of view of the theory of sound, is presented also, either in quite the same manner or in similar form, in several other questions of mathematical physics. We meet it when we consider other vibrations which are not acoustical—for instance, those that are electromagnetic. We meet it also in questions of another nature, such as those of the theory of heat.

A single result had been demonstrated rigorously since 1885, in such a way as to satisfy every mathematician. That was the analytic proof of the existence of the fundamental tone—that is to say, the one which corresponds to the absence of nodes and nodal lines in the vibrating membrane. Schwartz had obtained that result when studying certain questions of a different nature. For a long time he had been developing the theory of minimal surfaces—that is to say, the surfaces of equilibrium of a very thin liquid layer in which there is a surface tension (for instance, a layer of water in which soap is dissolved). In the problem of the calculus of variations, to which he was led, it was necessary to distinguish the maxima from the minima. He was thus led to consider the following question: A function of two variables vanishes at the boundary of a region of two dimen-

sions. The ratio of the value of its differential parameter of the second order to its own value is a negative constant at all points of the region. What is the smallest absolute value of this constant? Now the problem of the notes produced by the vibrations of the membrane consists in finding all the values of this ratio. That is why Schwartz's problem is only a particular case of the one we are considering.

The question then was to proceed to calculate all the other values beyond Schwartz's minimum. Already M. Picard had discovered properties of the greatest importance in this direction, and Poincaré had attacked the problem in a work which was published in the American "Journal of Mathematics," but it must be confessed that in this work he was still far from the solution. He took his revenge in the paper which we are about to examine.

We should guess from Lagrange's theorem and Lord Rayleigh's principle that the different pitches ought to appear as the roots of a transcendental function. It was the construction of one of these functions, or, more particularly, the proof of its existence, that Poincaré attempted. Let us see how.

He commences by adding a term to his equation—that is, he considers one that is made up of three terms. The first is the differential parameter of the second order, the second is the unknown function multiplied by a parameter, and the last is a function which he takes as arbitrary. We shall call this equation the auxiliary equation. The primitive equation lacked just this last term. He constructs this arbitrary function by linearly composing n functions by means of certain constant undetermined coefficients. This done, he develops the unknown function, supposed zero on the boundary, in a series of powers of the parameter. This result is reached by the use of Green's functions. He gets in this way an analytic function of the parameter for which

the development is valid within a certain circle, and which can be also represented as the ratio of two functions of which the denominator is independent of the variables of integration. By means of processes of extreme subtlety he shows that these undetermined coefficients of which we have just spoken can be chosen so that the two functions shall be entire functions of the parameter. Hence if in the auxiliary equation we replace the unknown function by the ratio of the two functions, giving them this entire form, we see that for all the values of the parameter which make the denominator vanish the auxiliary equation reduces to the primitive equation, and thus it comes about that all the roots of the entire function which appears as the denominator give the values which we were looking for.

Nothing can be simpler than this process which I have been able to summarize in so few words, but it contains a group of thoughts of a marvelous subtlety and fruitfulness.

What I have given is only the first part of Poincaré's memoir. The study of the roots of the functions which resolved the primitive equation, their properties, the developments that they followed, the definite applications to the problems of acoustics and of the theory of heat, give a number of very important results. They have been applied to many similar questions. At present this classic memoir remains as one of the finest monuments constructed by Poincaré; but it is with the methods of integral equations that we now study those problems. Leaving these questions here, however, let us pass on to other problems and investigations.

Some years ago it seemed for a time as if the atomic and corpuscular theories were losing ground. People thought that everything could be explained by means of continuous substances. In mathematical physics partial differential equations were obtained by abandoning entirely the molecular hypothesis. In chemistry also it was heard that the atoms

were becoming useless. But a sudden breath dispersed the light clouds which seemed to obscure the corpuscular theories. They are now supreme, and serve to illuminate the various regions of natural philosophy.

Necessarily the old atomic theories continued to advance. Electricity was first recognized as being of corpuscular nature, and little by little in every subject new sorts of atoms appeared. People discovered facts that accorded with the new theories. These theories became even the richest and most fruitful source of new discoveries, and it is for that reason that their reputation has increased from day to day. It has become now so secure that when contradictions are unavoidably presented we do not think of giving up these new ideas, but, rather, have not hesitated to abandon ancient principles whose validity was not doubted enough even to discuss. Little by little the classic theories which seemed set upon eternal foundations have been upset. Even mechanics, which, after Galileo and Newton, came to be regarded as the most secure of all sciences, has been overturned. A new mechanics has been formed, that of relativity. But that perhaps is already to-day an old mechanics. Will there not come from it indeed again an entirely new one, by virtue of the concept of atoms of energy?

Poincaré was associated with the transformation of the old physics and the birth of the new. His criticism and analysis have penetrated modern conceptions from all sides. He was devoted to such questions up to the end of his life, and several of his articles and latest lectures were given to their exposition. And so Poincaré was not only a master in the first kind of mathematical physics, but also in the second.

The electrodynamics of bodies at rest did not present great difficulties after the discoveries of Maxwell and the progress due to Hertz. But that of bodies in motion gave

rise to much discussion. Hertz had suggested a special hypothesis in order to pass from the case of rest to that of motion, but experiment proved it to be false, and it is Lorentz's theory which now explains best the latter subject. The celebrated discovery of Zeeman was a great triumph for the conceptions and hypotheses of Lorentz, because these conceptions and hypotheses predicted the doubling of the lines of the spectrum in a magnetic field; and this was the result verified by Zeeman's experiment.

Lorentz's theory was the source of a new order of ideas, including that which I have called the new mechanics. His theory was put in comparison with the principles of mechanics and physics. No contradiction appears with the principles of the conservation of energy or with those of electricity and magnetism. But at the first step a question is suggested to us, namely: Is it possible to determine explicitly the "absolute" motion of bodies, or, rather, their motion relative to the æther, by means of optical or electromagnetic phenomena?

To make the question still more precise: Do optical or electromagnetic phenomena serve to determine the absolute motion of the earth?

If we take account only of the first power of the aberration, the motion of the earth has no influence on any of these phenomena. This negative result has been shown by experiment, and is perfectly explained by Lorentz's theory.

But a celebrated experiment was performed by Michelson and Morley which kept account of the terms depending on the square of the aberration, and even this experiment, as is well known, gave a negative result.

In a famous paper of 1904 Lorentz showed that this result could be explained by introducing the hypothesis that all bodies are subjected to a contraction in the direction of the motion of the earth.

This paper was the point of departure for the later investigations. The results of Poincaré, Einstein and Minkowski followed closely that of Lorentz. In 1905 Poincaré published a summary of his ideas in the "Comptes Rendus" of the French Academy of Sciences. An extended memoir on the same subject appeared shortly afterwards in the "Rendiconti" of Palermo.

The basic idea in this set of investigations is founded upon the principle that no experiment could show any absolute motion of the earth. That is what is called the *Postulate of Relativity*. Lorentz showed that certain transformations, called now by his name, do not change the equations that hold for an electromagnetic medium; two systems, one at rest, the other in motion, are thus the exact images each of the other, in such a way that we can give every system a motion of translation without affecting any of the apparent phenomena.

In Lorentz's theory a spherical electron in motion takes the form of an oblate spheroid, two of its axes remaining constant. Poincaré found the particular force necessary to explain both the contraction of the electron and the constancy of the two axes. This is a constant exterior pressure acting upon the deformable and compressible electron. The work performed by this force is proportional to the variation in volume of the electron. In this way, if inertia and all of the forces are of electromagnetic origin, the postulate of relativity can be rigorously established.

But according to Lorentz all forces, no matter what may be their origin, are affected by his transformation in the same way as the electromagnetic forces. What modifications will it be necessary to introduce into the laws of gravitation, in virtue of this hypothesis?

Poincaré finds that gravitation must be propagated with the velocity of light. We might think, knowing the famous

theory of Laplace, that that was in contradiction with astronomical observations. But that is not so; there is a compensating term which removes every contradiction. Poincaré was thus led to propose and resolve the following question: To find a law which satisfies the condition of Lorentz and reduces to Newton's law when the squares of the velocities of the stars are negligible in comparison with the velocity of light.

Those are the fundamental problems and ideas of Poincaré, which have played such an important part in all later researches. The methods employed involve the principle of least action and the theory of groups of transformations, because Poincaré finds that the transformations of Lorentz form a group in Lie's sense. It is enough to have recalled these general ideas. At the present time they are much spoken of. They form the subject of such a great number of scientific papers and popular conferences that everybody knows them and appreciates their importance.

We shall close by speaking of Poincaré's contribution to mechanics. It is the hardest part of his work to analyze. He concerned himself with practically every branch of analytical mechanics: problems of stability, celestial mechanics, hydrodynamics and potential. The problem of the three bodies forms the subject of a great number of his investigations, now become famous, since they aided in revolutionizing classical methods. As is well known, it was Poincaré's memoir on "The Three-body Problem and the Equations of Dynamics" which was crowned with the prize founded in 1889 by King Oscar of Sweden. Important works of Poincaré's followed this memoir: the three volumes entitled "Les méthodes nouvelles de la mécaniques céleste," and the "Leçons" given at the Sorbonne. Moreover, Poincaré's last expository work was devoted to the discussion of the various cosmogonic hypotheses.

The fundamental ideas which guided Poincaré in the problems of mathematical astronomy were the consideration of periodic solutions, the study of the series which give the solution of the problem of three bodies, and the introduction of integral invariants. We have a periodic solution of the problem of three bodies if at the end of a certain time the three bodies are found again in the same relative positions, the whole system being merely turned through a certain angle. By considering the eccentricities and inclinations of the orbits, Poincaré was led to distinguish three kinds of periodic solutions for values of the time infinitely great either negatively or positively.

These studies on periodic solutions have very great theoretical interest, but also they have important practical applications. At a first glance, we can understand that the probability is infinitely small that in any practical problem the initial conditions of the motion will be such as to correspond to a periodic solution. Nevertheless, we can take one of these periodic solutions as a starting-point for a series of successive approximations, and thus study those solutions which differ little from it.

It is well known that a beautiful application of this method was made by Hill to the theory of the moon's motion.

The question of divergence of the series which appear in celestial mechanics has great importance. It is one of the most interesting questions that have arisen in mathematics. Can we use divergent series, and can we by means of series of this kind arrive at approximate solutions of practical problems? The example of Stirling's series allows us to answer in the affirmative. We find series of the same kind in celestial mechanics. They also furnish approximate values sufficient for the demands of practice. That is what Poincaré noticed and proved.

The celebrated theorem about the non-existence of uni-

form integrals—that is to say, that the three-body problem has no uniform integrals besides those already known—is one of the most remarkable results of Poincaré's theory.

In these researches about which we have been speaking the so-called integral invariants play an essential part. These are approximations which are calculated by quadratures applied to the variables of differential equations, and remain constant. These invariants are connected intimately with the fundamental question of stability.

It is impossible to summarize all these theories and yet present them clearly. On the other hand, to develop them more minutely would carry us too far.

Following the same path that we have taken for analysis and mathematical physics, let us then consider also in mechanics a particular one of Poincaré's investigations, sufficient to show us the range and powerful originality of his genius. On the one hand, this investigation is related to hydrodynamics; and on the other, to celebrated questions of celestial mechanics and, as Sir George Darwin has shown, to the most interesting and modern cosmogonic theories.

It is the question of the equilibrium of a rotating fluid mass, and was one of the first problems that presented themselves with the establishment of the theory of gravitation. MacLaurin gave a solution of it by means of ellipsoids of revolution, and it is perhaps the finest result which that great geometer gave to science. The solution by Jacobi by means of ellipsoids with three unequal axes was a happy stroke of genius of that illustrious mathematician. He was in fact the first to doubt what everybody considered as evident *a priori*—that is, that every possible form of equilibrium of a rotating homogeneous fluid mass is symmetric in regard to the axis of rotation.

But solutions due to MacLaurin and Jacobi were only particular solutions of the general problem. There are an

infinite number besides. We must also notice that these particular solutions were not obtained directly. It was merely verified that under certain conditions certain ellipsoids satisfied the laws of equilibrium.

Before considering Poincaré's investigation we must recall the fact that Thomson and Tait in their treatise on natural philosophy had seen that there were ring forms of equilibrium as well as ellipsoids. They had also studied the question of stability, either by imposing certain conditions on the fluid mass—for instance, that of being a solid of revolution or of being ellipsoidal—or by omitting such conditions.

The fruitful idea of Poincaré was that of equilibrium of bifurcation. Let us consider a system whose state depends on a certain parameter. If, for instance, we have a rotating fluid mass, we can let that parameter be the angular velocity of rotation. Let us suppose that several different forms of equilibrium correspond to the same value of the parameter. Let us change that value. The configurations—or, in other words, the forms of equilibrium—will change. It may happen, that, on approaching a certain limit, two forms of equilibrium become the same. If we go by this limit we may have one of two cases. The figures of equilibrium may disappear; we express this in algebraic language by saying that they become imaginary. That is the first case. We say then that that form which the two figures approach is a limiting form. But it may happen that if we pass the limiting value the two distinct figures reappear. That is the second case. In this case the figure where the two forms of equilibrium coincide is called a *form of bifurcation.*

Let us suppose ourselves to be able to represent each figure of equilibrium by a point in the plane of which the coördinates are the value of the parameter and some vari-

able which distinguishes the figure. By changing the parameter we shall have a curve. In our second case this curve is formed of two branches which cross, corresponding at their intersection to the form of bifurcation. Now Poincaré established a theorem of the greatest importance by considering the stability of the figures corresponding to the different points of the two branches. Let O be the value of the parameter which refers to the point of intersection. If for negative values of the parameter there is stability on the first branch and instability on the second, it will be the opposite for positive values of the parameter—that is, there will be instability on the first branch and stability on the other. In other words, there is an exchange of stabilities between the two branches at the place where they cross. This proposition was called by Poincaré the *theorem of the exchange of stabilities.*

Let us now apply these results to the question of the rotation of fluid masses. Let us suppose that we know the solutions of MacLaurin and Jacobi. The axis of rotation is always the small axis of the ellipsoid, and so we know that its ratios to the other axes are less than unity. These ratios are equal for MacLaurin's ellipsoid and different for that of Jacobi. If

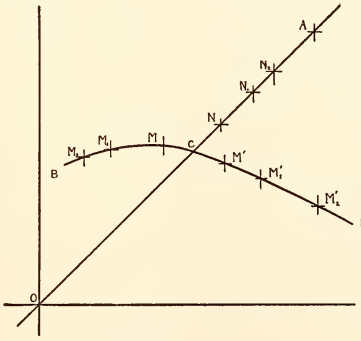


FIGURE 1

we take these ratios as coördinates of a point in the plane, each ellipsoid will be characterized by a point, and these points will form a curve (see Fig. 1). The bisector of the angle between the axes will be the line that represents the ellipsoids of MacLaurin. The curve BCD will represent the ellipsoids of Jacobi.

But Poincaré also found new figures of equilibrium that can be obtained by deforming the ellipsoids. The exact form can be calculated by means of Lamé's functions. The simplest have the form of a pear. It is shown that there exist an infinite number of ellipsoids of MacLaurin that correspond to the points N, N_1, N_2 . . . . of the line AO such that one of the infinitely near figures of Poincaré is also a form of equilibrium. In the same way there are an infinite number of points M, M_1, M_2 . . . . M^1, M^1_1, M^1_2 . . . . of the curve BCD such that the neighboring Poincaré figure is also a form of equilibrium.

Let us consider now the stability. The MacLaurin ellipsoids are stable in the part AC, and unstable in the part CO. The ellipsoids of Jacobi are stable from C up to the first point M, where one encounters a figure of Poincaré, and unstable in the part MB.

Hereupon we come to an application of the theory. I quote from Poincaré himself:

"Let us consider a homogeneous fluid initially rotating and cooling slowly. If the cooling is slow enough the internal friction determines that the whole mass revolve with the same angular velocity at all points. The moment of rotation will moreover remain constant.

"At the beginning, since the density is very small, the form of the mass is an ellipsoid which will hold together despite the revolution. The representative point will describe the portion AC of the line which corresponds to the MacLaurin

ellipsoids up as far as C, where these ellipsoids become unstable. The representative point, which can no longer take the path CO, will then follow, for instance, the direction CM; the ellipsoid will have its three axes unequal, and this is true as far as M, where the Jacobi ellipsoids become unstable. Beyond this stage, since the mass can no longer keep the ellipsoidal form, that having become unstable, it will take on the only form possible, which is that of the neighboring surface to it. This surface is a piriform figure (see Fig. 2) which has a narrow place in the region marked 3; the regions 2 and 4 tend to increase at the expense of the regions 1 and 3, as if the mass were trying to divide in two unequal parts."

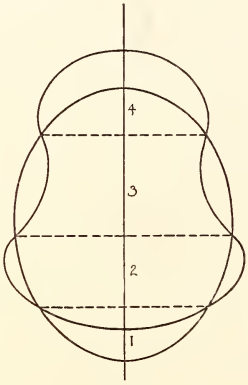


FIGURE 2

The results that we have just presented are quite elegant and of great importance. They revealed much to Sir George Darwin. He thought that the process which we have just described might play a part in the evolution of celestial systems, and this theory seems to be confirmed according to the forms observable in many nebulæ. Some satellites may have been formed in this way at the expense of their planets. In particular that may have happened in the case of the earth and the moon, the masses of which are comparable in magnitude.

Under these majestic aspects, where the most subtle and ingenious theories of mechanics are at one with the most

daring cosmogonic hypotheses, we finish our analysis of the investigations due to Poincaré.

I have given but an incomplete idea of the immense work which he did, of the problems which he treated and which it will be necessary to study exhaustively, of the regions which he has opened where several generations of mathematicians will be able to work.

His discoveries will but have the result of stimulating new investigations. That is the fate of the works of great geniuses. They give the key for solving many problems and satisfy scientific curiosity by unveiling the secrets of nature, but at bottom they merely increase that curiosity by opening new horizons and making still more distant the goal of scientific aspiration.

VITO VOLTERRA.

William Ramsay.

THE ELECTRON AS AN ELEMENT
COMPOUNDS OF ELECTRONS
THE DISRUPTION OF THE SO-CALLED ELEMENTS[1]

First Lecture

THE ELECTRON AS AN ELEMENT

THE independent existence of the electron is now conclusively demonstrated; in my opinion, it is, next to the first and second laws of energy, the most far-reaching discovery which has yet been made, both in its application to the elucidating of our former views concerning matter and its nature, and to our control over what are popularly termed "the forces of nature."

Although progress in human thought has usually been achieved from the practical standpoint, still, after a sufficient number of observations have been made, a consistent theory, which permits of the knitting together of such isolated parts into a complete whole, suggests the trend of further research, and renders easy what previously was justly regarded as difficult. It is thus with the idea of the electron as an entity. Once it has been realized that the part played by the electron is all-pervading; that it enters as an element into the constitution of chemical compounds; that when they undergo

[1] Three lectures delivered at the inauguration of the Rice Institute, by Sir William Ramsay, K.C.B., F.R.S., Professor of Chemistry in the University of London.

change, that change is brought about by a shifting of electrons from one form of combination to another; when we realize that a current of electricity flowing along a wire is merely the passage of almost infinitely numerous electrons from place to place, and the formation and decomposition of temporary compounds; when we can clearly conceive that the starting and stopping of such a current of electrons cause ethereal waves, themselves capable of starting or stopping similar currents of electrons in wires parallel to the first; when we realize that by the expenditure of energy such streams of electrons can be set in motion and can be stopped,—then we have acquired knowledge which will enable us to contrive machines better than those which we already possess, whereby the direction of motion of electrons can be controlled.

The fact that electrons cannot be seen need now prove no stumbling-block. For men were for long unable to realize that invisible gases could be put to use. The wind was by our forerunners regarded as semi-spiritual; a ghost and a gust were akin; and I find it difficult to convince my non-scientific, and even some of my scientific, friends that it is much easier to work with and to manipulate gases than liquids or solids. And now gases, in the form of compressed air, compressed steam, or compressed products of explosion, are our chief agents for conveying energy from place to place; they are, electrons excepted, the means by which almost all our energy is transmitted. They have the advantage of being easily moved; of being elastic; and of being conveyed rapidly from place to place without loss. Indeed, if it were necessary to characterize the past century by a single expression, the "age of compressed gases" might be aptly chosen.

The story of the measurement of the mass of an electron has often been told. The "kathode rays" were discovered by Lenard to be able to pass through a thin sheet of alu-

minium; and after their passage they were found to be able to penetrate the atmosphere for some distance, although they were somewhat rapidly dispersed; in fact, the dispersion, with the loss of their activity, has been likened to the passage of light through water to which a few drops of milk have been added. Crookes's previous researches had proved that kathode rays can be concentrated to a point from an aluminium kathode, shaped like a parabolic mirror; that they produce great rise of temperature at their focus; that their impact can impart rotatory motion to a paddle-wheel on the blades of which they impinge; and that they have the property of causing phosphorescence in various objects—many minerals, for example, glowing with marvelously brilliant colors. They are unable to penetrate thick objects; hence a metal cross or other object can be made to cast a kathode shadow when placed in their path, and the shadow can be well seen on the side of the glass vessel in which the rays are generated; the glass phosphoresces with a beautiful green or blue color, except where it receives the shadow of the metal cross.

Crookes also showed that two such streams of electrodes, each arising from its own kathode, repel each other; for if the kathodes were parallel, the streams were not parallel, but divergent. On the other hand, streams of kathode particles, passing in opposite directions, attract each other.

Goldschmidt, many years ago, had noticed that the streams of kathode rays can be deflected by a magnet; and it was this property of the rays, taken with that of their being attracted by a positive and repelled by a negative electric field, which led to the possibility of measuring the ratio of the charge which they carry to the mass of the electron.

Knowing this ratio, it follows that if the magnitude of the charge be known, the mass of the electron will then be deter-

mined. Now, accurate measurements show that this ratio involves one of two alternative suppositions: either that the negative charge is 1830 times the positive charge carried by one atom of hydrogen in the ionic state, or that the mass of the particle is only $\frac{1}{1830}$ of that of an atom of hydrogen. It appeared improbable that the first supposition should be correct; and the matter has been decided without a shadow of doubt from experiments made by Mr. C. T. R. Wilson. A property of ions in a gas is to cause the condensation of supersaturated water-vapor to droplets. The number of such droplets can be counted; the velocity of their fall can be measured. This affords a means of determining the diameter of each droplet, and from that the volume of a droplet can be deduced; and as the total quantity of electricity carried down by the precipitated liquid can be easily measured, the charge on each particle can be estimated. It is that which may be attached to one, two or more electrons; for the ion of a gas may be attached to electrons, and each ion corresponds to one water droplet. Wilson's experiments, as well as the beautiful experiments of Milliken, agree in the conclusion that the electric value of a unit charge, or electron, is 4.78×10^{-10} electrostatic units; and it follows from this that the mass of an electron is $\frac{1}{1830}$ of that of an atom of hydrogen.

It is possible now to go further and to determine the actual mass of an electron. Experiments by M. Perrin on what may be termed visible molecules—namely, particles of gamboge in an aqueous emulsion—have enabled him to deduce with great accuracy the mass of an atom of hydrogen; it is 1.63×10^{-24} gram. Dividing by 1830, the mass of an electron is found; it is 0.8×10^{-27} gram.

Let me interpose here the remark that the method of determining the "atomic weight" of an electron does not differ in principle from the usual method of determining

atomic weights. The usual method is to ascertain the weight of the element in question which will combine with a known weight of some standard element the ratio of whose atomic weight to that of oxygen is known. This ratio is generally determined by the balance, and the result gives the equivalent of the element of which the atomic weight is required. With the electron the process is similar, except in the method of weighing; the "weight" is determined electrically. Indeed, the use of the word "weight" is not strictly permissible, for the attraction of the earth does not come into play; electric forces replace it. But there is now no doubt that the atomic mass of an electron is $\frac{1}{1830}$ of that of hydrogen. It is also certain that what is termed an electric current consists of a stream of such electric particles in motion; and that a negative electric charge consists in the surface of the negatively electrified object being covered with a film of such particles.

We see, therefore, that we have now to do with an element of known atomic weight which has been isolated from its compounds and is thus accessible in the free state. It may be pointed out here that this is not the first time that the existence of elements has been inferred before their isolation in a state of freedom. To quote a familiar instance, fluorine was defined as an element by Davy eighty years before Moissan prepared it by electrolysis of hydrogen fluoride, rendered a conductor by the presence of dissolved salts. The fact of the general resemblance of its compounds to those of the other halogens made the inference legitimate. But the electron possesses properties so remarkable that there is little wonder that its elementary nature was overlooked.

The first suggestion, which, nevertheless, fell short of the truth, was made in 1887 by Helmholtz in his Faraday lecture, when, having indicated that according to Faraday's law

each atom of an element, liberated on electrolysis, is associated with one or more units of positive or negative electric charge, he pointed out that the legitimate conclusion to be drawn was that each liberated elementary atom is associated with one or more positive or negative units of electricity, to which the term "electric atom" might legitimately be attached. It has only been slowly realized that a negative charge is due to the presence of atoms of electricity, or negative electrons, and that a positive charge is due to their absence. We are reminded by this of the long-exploded doctrine of phlogiston, the demolition of which by Lavoisier revolutionized the science of chemistry and gave it a fresh start. In it the absence of oxygen corresponded with the presence of phlogiston, a wholly imaginary conception; just as a positive charge was tacitly assumed to be the addition of positive electricity to matter, while a negative charge corresponded to the association of matter with negative electricity. It is as if the upholders of the phlogistic theory, having been convinced against their will that combustion implied combination with oxygen, had at the same time maintained that during such combination phlogiston is lost. Indeed, Scheele's ingenuity made him devise a somewhat similar hypothesis when he was confronted by the experimental fact that oxygen is produced by heating "mercurius precipitatus per se" in a retort. His explanation was that the heat which entered the retort, being composed of phlogiston plus fire-air, was decomposed by the calx of mercury; the calx, combined with the phlogiston, producing mercury, while the fire-air, or oxygen, the other component of "heat," escaped and could be collected. The reasoning is perfect as long as the use of a balance is excluded; and, as with the electron, it was only by careful weighing that the substantiality of oxygen could be demonstrated.

Similarly, it is now time to reject the old hypothesis that there are two kinds of electric fluid—one positive, one negative; the evidence is overwhelmingly in favor of the theory that electricity consists of an assemblage of electrons, or particles of negative electricity, and that compounds of electrons change their nature when the electrons are removed, just as mercuric oxide acquires the properties of a metal by removal of oxygen. Much confusion has arisen owing to the fact that electric phenomena are produced by ethereal waves. Indeed, the word "electricity" has a dual signification: firstly, it applies to congeries of negative electrons attached to what is generally termed matter, as one element is united to another—or, to use a more general expression, is attached to another, or to a compound; and secondly, it is made to signify vibrations in the ether, which arise when a current of moving electrons is started or stopped. It is also clear that a magnet is associated with electrons in circular motion, which keep the neighboring ether in a state of strain; if the lines of strain, or "lines of force," be cut by a moving wire, the electrons in that wire are set in motion and a current is produced. It is unnecessary to state that this fact that ethereal vibrations can start or stop electrons has proved of the very greatest service to mankind; to this is due the invention of the dynamo, of the motor, and of wireless telegraphy. But it is evident that such ethereal vibrations, transmitted as waves, are in no sense the material electrons, any more than the force applied by a horse to a rope is the canal-boat which it sets in motion.

As for the mechanism by which ethereal waves effect motion in electrons, that is beyond the scope of these lectures. Indeed, of the rival theories which profess to explain it, not one is satisfactory. All that can definitely be said is that there is an evident gyroscopic action, for motion of

electrons occurs not in the direction of propagation of the ethereal force, but at right angles to it. We therefore deliberately confine our attention to the electron as a form of matter with a known atomic weight, viz., $\frac{1}{1830}$, and capable of forming compounds with what we commonly term matter. And here again we must draw a line. The question has been raised, Does matter consist of congeries of electrons in rotation, or in vibration, or exercising some form of relative motion? Or is there a material nucleus, composed of some entity different from electrons, with which electrons can combine, and from which they can separate? And is there only one such stuff—primordial matter? Or are there as many varieties of stuff as there are elements?

These speculations are of great interest; some of them have exercised men's minds for centuries. But answers to these questions are not yet forthcoming; they are the goals to which investigation is tending. As regards the question of the composition of matter, whether it consists wholly of electrons or not, that must be left open. It can and will be decided by experiments devised to test various theories. All we need say for the present is that most forms of matter, such as we know them, contain electrons as parts of their composition; we need not *yet* concern ourselves with the constitution of the residual matter after the removable electrons have been removed.

As for the unity of matter, I hope to be able to show that progress is being made in the direction of an answer to that question. It may, however, be stated at once that it is as yet absolutely uncertain whether or not matter will ultimately be found to be homogeneous—that is, consisting wholly of one kind, associated with more or with fewer electrons.

Having arrived, then, at the notion that in electrons we must recognize an elementary form of matter, let us next

consider the transference of electrons from one form of combination to another. This can be done most simply by reasoning on any simple electrolysis; and I will choose that of water, assuming, for simplicity's sake, that the change is the theoretical one, $2H_2O = 2H_2 + O_2$. The real change which occurs depends, of course, on the electrolyte which has been added to the water, and on the action of its liberated ions on the water; if it be sulphuric acid, for example, the hydrogen of the acid will be set free, and the sulphation group, SO_4, will liberate oxygen by its action on water. We will neglect these actions, however, and will regard the action as expressed by the simpler equation.

Water, then, consists of molecules of some complexity, probably H_6O_3, or H_8O_4, or mixtures of these with even more complicated molecular groups; and along with them, mingled with the rest, are ions of hydrogen and oxygen. The hydrogen ions are those which lost electrons to the oxygen when the water was produced. It is reasonable to suppose that during the combination of the hydrogen gas with the oxygen gas (granting the water to have been so formed), the hydrogen, which as a gas consists of hydrions in union with electrons, H–H–, has, during its "union" with oxygen, which as a gas may be provisionally taken as O=O, given its electrons to the oxygen; so that on ionization the electrons, having already arranged themselves in the water-molecule in such a manner that they are no longer directly associated with the hydrogen, leave the hydrogen atoms entirely without removable electrons; it is often the custom to call these atoms of hydrogen devoid of electrons, "hydrions." Each electron which has left an atom of hydrogen associates itself with an atom of oxygen plus one of hydrogen, thus:

$$2H--H+O==O = 2H-O_{\equiv}+H+H.$$

This equation requires consideration. A molecule of hydrogen is not H–, but $=H_2$. Now it is an open question how the electrons are attached; but it is to be presumed that an electron forms the bond between the two atoms. This may happen in two ways. First, the attachment may be H––; or, second, H–H–. The same reasoning applies to the molecule of oxygen; it may be =O=O or O==O. In the first case one of the atoms is tetrad, according to the usual code of writing; but that need excite no surprise: oxygen is known to possess tetrad valency under suitable conditions. It may be remarked, however, that similar reasoning applied to the hydrogen molecule involves the assumption of dyad hydrogen, and that is an unlikely supposition. It need hardly here be insisted on that the actual practical valency of an element or group is equal to the number of electrons which it carries during electrolysis; that is the corollary of Faraday's law. Now hydrogen is invariably monovalent; hence the formula H––H is preferable to H–H–. On the other hand, it may be objected that two electrons will repel each other, and it might with justice be asserted that for that reason H–H– is preferable to H––H; and similarly that O=O= is preferable to O==O. This statement will be referred to again in the second lecture. Perhaps both formulæ are correct; tautomerism may occur in reference to atoms and electrons as well as between atoms considered independently of electrons; the formula of hydrocyanic acid appears to be both H–C≡N and H–N≡C; and many similar instances will suggest themselves in more definite cases, as, for example, among the enols.

Leaving such questions for the present, let us see the effect of an electric current on hydrions and hydroxylions. They are to be regarded as separate and definite chemical entities intermingled with complex water-molecules—indeed, sur-

rounded by them; for it is in every way probable that the hydrions are attracted by spare electrons of the water-molecules. We have many instances of a similar directive action among compounds; the place of substitution in the benzene ring depends on the position of groups already substituted for nuclear hydrogen. We may therefore believe that the ions both of hydrogen and of hydroxyl are protected by a coating of non-ionized molecules of water. It is, indeed, probable that interchange of electrons takes place between the two, molecules and ions, so that it is not always the same hydroxyl group which retains its electron; the Williamson-Clausius hypothesis of interchange may well be applicable.

Into such a system of molecules and ions two platinum electrodes are plunged. We need not here consider the source of the current; suffice it to say that at the negative electrode the electrons are crowded on the surface, ready to escape on application of sufficient driving force—*i.e.*, of a sufficiently high potential; while from the positive electrode the electrons are subject to strain, for they are being sucked into the connecting wire by a corresponding electromotive force. In fact, we may consider the negative electrode as a region of electric pressure—a kind of electric force-pump; and the positive electrode as a partial electric vacuum—an electric suction-pump.

The hydrions, having no electrons attached to them, are attracted to the negative electrode, where electrons are present under electric pressure; they move thither at a rate depending on the mobility of the ion (and hydrions are the most mobile of all ions) as well as on the viscosity of the liquid, which is itself a function of temperature. Having arrived at the kathode, each ion absorbs an electron, and from a hydrion becomes an atom of hydrogen. Each atom

of hydrogen readjusts its newly found electrons so as to combine with its neighbor atom according to one of the schemes already set forth.

In an exactly similar manner, the kation, the hydroxylion, reaches the anode where electrons are under strain; from each hydroxylion an electron is removed, and the group OH is left without a free valency—*i.e.*, without an attached electron. It may under certain circumstances unite with another hydroxyl group, due possibly to the quadrivalence of the oxygen atoms; they may serve as bonds of attachment of the two groups to each other, thus:

$$O\text{–}H+O\text{–}H = H\text{–}O \equiv \equiv O\text{–}H;$$

or only one of the three latent electrons may come into play, thus forming

$$H\text{–}O\text{--}O\text{–}H,$$

the others being existent, though not in evidence. Or, as more generally happens, the molecules readjust themselves, forming water and free oxygen according to the scheme

$$O\text{–}H+O\text{–}H = H\text{–}O\text{–}H+O;$$

and the atom O unites with a neighbor atom of O, forming

$$O{=}{=}O \text{ or } {=}O{=}O,$$

as explained before.

It may be objected that views such as the above are very hypothetical; that they tend to complexity and not to simplicity; and that they are imperfect. To that it may be replied that it is certain that some ions are carriers of electrons, and that others—the positive ions—travel without manifest electrons; that the electron is certainly to be regarded as an element, and that its comings and goings, its entering and escaping from chemical compounds must therefore be chronicled in all complete equations; that the intro-

duction of a new element capable of reacting with other elements necessarily tends toward complexity; and that all first attempts to represent chemical changes are of necessity imperfect, as is witnessed by the enormous progress which has been made in the graphic notation of organic chemistry.

This example will serve to illustrate the electrolysis of any chemical compound; the processes which occur are similar in kind, although they may differ according to the nature of the electrolysis.

Let us next consider what goes on in a simple battery; and we may suppose a plate of platinum and a plate of zinc dipped in a bath of dilute hydrochloric acid and coupled by means of a wire, a galvanometer being inserted to show the direction and electromotive force of the current.

The solution contains chlorions and hydrions, each protected by water-molecules. The more dilute the solution, the more efficient the protection from mutual discharge of the anions and the kations, the greater the ionization of the solution. Concentration of the solution by diminishing the relative number of water-molecules decreases the number of ions of hydrogen and chlorine. These ions are to be supposed, before introduction of the platinum and zinc plates, as evenly distributed throughout the liquid.

The plates are now introduced but not yet joined by a wire. Now, zinc, for some reason which we cannot yet guess at, has a greater tendency to dissolve in water than has platinum. But metallic zinc, which is really a compound of a zincion with two electrons, is insoluble in water; to dissolve, it must lose its electrons. When placed in water which contains some few hydrions, a trace of zinc will doubtless dissolve as ions, while a trace of hydrogen will adhere to the surface of the zinc. But the pressure—the solution-pressure, as it may be termed—will soon cease, and no further action

will occur. On joining the zinc plate to the platinum plate by means of a wire (let us suppose of copper), the zinc begins to dissolve, while for every atom of zinc dissolved a molecule of hydrogen attaches itself to the surface of the platinum, and when the concentration is sufficient it escapes in bubbles.

In order that the zinc shall dissolve it must lose its electrons. These, however, require a channel of escape, which they find in the copper wire. Leaving for a moment the nature of the change which accompanies their transit, let us follow them to the surface of the platinum plate. Here they accumulate, with a pressure (that is, at a potential) equal in absolute measure to the solution-pressure of the zinc plate. The hydrions flock to the platinum plate, for they, lacking electrons, travel to where electrons are plentiful; each hydrion acquires an electron, unites with it, and, as previously explained, joins to a neighboring atom to form a molecule. When these attain a sufficient number to saturate the neighboring water, and the capacity of platinum for holding atomic or molecular hydrogen (probably atomic) is attained, the molecules of hydrogen escape in bubbles.

The chlorions—in the old nomenclature negatively charged, in the new conception containing each an active electron—are attracted to the spot from which electrons are flowing away through the wire. Although they are not otherwise changed, they concentrate in the neighborhood of the anode, from which zincions are being propelled into the solution. The rate of their flow to the anode depends on their specific mobility and on the viscosity of the liquid, a condition of concentration and temperature.

In short, the process taking place in a battery has considerable resemblance to that which causes the flow of a liquid due to osmotic pressure. A concentrated solution, in contact

through a semipermeable diaphragm with a dilute solution, tends to be diluted; the solvent from the dilute solution passes through the semipermeable membrane into the concentrated solution, and lessens its concentration. Now the electrons may be likened to the solvent of the dilute solution; they have alternative courses. The wire is permeable to electrons, but not to ordinary forms of matter; it acts thus as a semipermeable membrane. The pressure may, as in the case of osmosis, be regarded from two points of view: either as that of the solvent entering the concentrated solution through the semipermeable membrane, or as due to the bombardment of the walls of the vessel containing the concentrated solution by the molecules of the contained solute. So the pressure in the battery may be regarded from two points of view: either as the difference between the solution-pressure of the metallic zinc and that of the metallic platinum, or as the difference in the affinity of electrons for zinc and for platinum. It is, however, the property of the electrons to pass along the wire, which differentiates them from what we generally term matter; and, as already remarked, the phenomena in a battery afford a close analogy with those producible by means of osmotic pressure. We have in the battery a stream of electrons passing along the copper wire as long as there is zinc to dissolve in the ionic state, or as long as ions of hydrogen remain in solution to unite with the electrons on the surface of the platinum. This current of electrons may be made use of in several ways; first, it may be employed in electrolyzing an interposed solution—that phenomenon has already been considered. Second, it may serve to heat the wire; the conditions for a great rise of temperature are that the wire shall be thin, and that its conductivity shall not be high. Third, if the wire be coiled, and

if a magnet be suspended within the coil, it will set itself at right angles to the plane of the coil.

Let us first consider the heating of the wire, for that involves the theory of metallic conduction.

All material elements are capable of combination with electrons. Those which are termed bad conductors or insulators, however, do not readily combine; the electrons therefore form a layer on the surface. Such a layer can be produced by friction between two non-conductors—for instance, silk and sulphur. As has been known for a century and a half, "frictional electricity" can be produced by rubbing a silk pad on a cylinder of sulphur. Here the surface of the sulphur is "negatively electrified"—*i.e.,* electrons leave the silk and adhere to the sulphur. If a glass cylinder be substituted for one of sulphur, it is the glass which loses electrons and the silk pad which gains; hence the old names "vitreous" for positive and "resinous" for negative electricity. The rubbing of a metal object also effects the transfer of electrons; but in this case, unless the metal is supported by a non-conductor, the loss or gain of electrons is replaced by conduction from the earth. The electrons spread themselves all over the surface of the metal, instead of adhering in patches, as they do if non-conductors be rubbed.

When a salt is dissolved in water it is the metallic portion which loses its electron or electrons, and the non-metallic portion which gains them. We may therefore conclude that metals have less tendency to combine with electrons than non-metals, and that the more "metallic" an element, the less its tendency to hold electrons. It is therefore to be expected that in a metal wire, if electrons are introduced at one end, they will displace those in combination with the metal at the hither end of the wire, and that this process will go on con-

tinuously, so that if it is possible for electrons to escape at the further end, they will pass from one end of the wire to the other. This will also happen with rods or wires of poor conductors, but not with actual non-conductors. Transparent fused salts, or oxides, such as rock-salt, glass, or silica, are practical non-conductors. Their only method of conduction is an electrolytic one, and the mobility of their molecules and ions is so small that they cannot serve to convey electrons. But in a copper wire the transfer of electrons is easily effected.

The result of the passage of a current through a poor conductor of small section is to heat it. This heat corresponds quantitatively with the resistance which it offers to the passage of the current. It may be conceived that the electrons form relatively stable compounds with the atoms of the element of which the resisting wire is composed, and that in order to facilitate their passage the atoms are obliged to readjust their position relatively to each other; hence friction and heat. It would follow that the electrons do not flow as a stream through the interstices between the atoms, but that they form temporary and unstable compounds with the metal as they flow. It must, however, be acknowledged that this explanation lacks completeness, which further experiment will doubtless assure.

It is somewhat beyond the scope of these lectures to consider the action of a stream of electrons on the position of a magnet. The flow of electrons, it may however be remarked, produces a strain in the ether which interferes with the rotation of the electrons round the atoms of a magnetized bar. These set themselves at right angles to the plane of the wire carrying the current. Conversely, a forcible displacement of the magnet will cause a shift of electrons in the wire. But, as before remarked, owing to lack of definite-

ness in our ideas of the nature of the ether, no perfect picture has yet been made of the mechanism of its action.

We are more and more impressed with the necessity of a mechanical conception of things around us; all recent discovery shows that things much too minute for us to see are constituted in a manner not unlike the objects apparent to our senses. Hence we must regard the atoms of electricity —the electrons—as capable of taking up position in a chemical compound, just as we have imagined the atoms to do. It is true that we cannot maintain that the atoms are without motion. Far from it. But we can with fair probability determine the position of their centers of oscillation or rotation. The structure of compounds, viewed from the electronic point of view, will form the subject of the next lecture.

Second Lecture

COMPOUNDS OF ELECTRONS

WE have already considered one compound—namely, water—from the point of view of the shift of electrons during the reaction of gaseous hydrogen and oxygen. It may conduce to clearness, however, if similar considerations are applied to the case of sodium chloride, one method of preparing which, though far from a commercial one, is the "direct union" of sodium with chlorine. It may be remarked, however, that union does not take place between perfectly dry chlorine and clean sodium; it appears to be necessary that a trace of water-vapor be present. The rôle played by the water will be considered later.

On the electronic hypothesis, sodium, the metal as we know it, is a compound of an atom of sodium with an electron. Chlorine, too, is a compound of an atom of chlorion with electrons; and inasmuch as in the perchlorates chlorine functions as a heptad, it would perhaps be proper to indicate that fact whenever the chlorine symbol is written. A convenient method is to affix to the symbol of the element the Roman numeral VII; thus, Cl^{VII}. These electrons, however, so far as we know, play no part in the union of sodium with chlorine; hence it is permissible to omit them for the present. It is, however, to be noted that the addition of one more electron to chlorine raises the total number of attached electrons to eight; and this appears to be the highest number of electrons with which an element can be associated.

The equation $Na-+Cl=Na-Cl$ does not accurately express the whole change, for that is undoubtedly preceded

by the change $Cl{-}{-}Cl = Cl{-}{+}Cl{-}$, and the simple atoms of chlorine are available for combination. It is, of course, possible that the chlorine molecule persists by reason of the interaction of several electrons; then Cl^{VII} may be attached to its neighbor Cl^{VII} by many "bonds"; $Cl^{VI}{-}{-}Cl^{VI}$ would express the case already mentioned; but $Cl^{V}{=}{=}Cl^{V}$ or $Cl^{IV}{\equiv}{\equiv}Cl^{IV}$, etc., would equally well represent combinations of the sort. In the absence of any positive evidence, the simplest hypothesis may be adopted, and the abbreviated form $Cl{-}{-}Cl$ chosen.

Union between an atom of sodium and one of chlorine consists, in all probability, in the use of the electron of the sodium. The formula $Na{-}Cl^{VII}$ would appear to represent the compound, because as soon as that salt is dissolved in water the electron is undoubtedly associated with the chlorine atom; we have Na, surrounded by water-molecules on the one hand, and on the other, $-Cl^{VII}.Aq$. Now sodium chloride does not differ in properties from its solution, except in so far as the ions are free to migrate after, but not before, it is dissolved. The salt has a specific refractivity; its solution possesses a refractivity practically the mean of that of the salt and the water, taken in the proportions in which they are present in solution. It has also a mean specific heat, and, in short, many other physical properties of the same order. It is therefore more than probable, if the existence of electrons be granted at all, that the change in position of the electron originally attached to the atom of metallic sodium has taken place during the formation of the sodium chloride. And on solution in water, the new system divides: the sodium ion, surrounded by attracted water-molecules, constitutes one practically independent unit, Na.Aq; while the chlorine ion, $-Cl^{VII}Aq$, has also reached independent existence. This is proved by the fact that these entities

exert each its own calculated osmotic pressure; and furthermore, that the chlorion can be attracted to a metallic anode, and the sodion to a metallic kathode, placed in the solution. Similar reasoning may be applied to the ordinary hydroxides of the metals, even to those which are ordinarily regarded as insoluble; for insolubility is only a relative term, and reactions between hydroxides and acids are no doubt only effective as regards that portion of the hydroxide in solution; because, when withdrawn, and after reaction with the acid to form a salt and water, it is at once and continuously replaced, according to well known laws, by a further portion which goes into solution as ions of metal and acid-radical. It may be imagined that the attack of a metal like sodium by chlorine, which depends on the presence of a trace of moisture, has also to do with the action of the metal on the water. It is, however, not so easy to give a reason. For the loss of energy due to direct formation of salt from sodium and chlorine is obviously, according to Hess's law, the same as that which ensues when salt is formed indirectly, according to the usual scheme $2Na + 2HOH = 2NaOH + H_2$, $H_2 + Cl_2 = 2HCl$, and $2NaOH + 2HCl = 2NaCl + 2HOH$; the molecule of water being regenerated. It may be that such a system permits of the easier transfer of the electrons; this answer, however, begs the question; or it may be that either the chlorine or the sodium, or both, enter into combination with the water-molecules, making use of the latent electrons of the oxygen, thus: $H_2{=}O{=}Cl_2^{VII}$ and $H_2{=}O{=}Na_2$, and that these subsequently interact with one another. This, however, opens a question, afterward to be considered, relating to the source of the electrons, which are depicted as bonds between the oxygen and the chlorine on the one hand, and the oxygen and the sodium on the other.

The case of salts in general is analogous to that of the

chlorides and hydroxides. The "acid radical" is in itself an ion, carrying with it, according to its basicity, mono-, di-, tri-, etc., one, two or three electrons. Thus the group $=SO_4$ has doubtless two available electrons; $\equiv PO_4$, three, and so on. The portion of those salts, generally regarded as insoluble, which is in a state of solution contains such ions; and, indeed, a determination of the conductivity of the very sparingly soluble salts affords an elegant plan of determining their solubility.

In certain cases ionization does not occur so simply as to be represented by a metallic anion and a non-metallic kation. Compounds such, for example, as cupric chloride ionize, at least partially, into Cu.Aq and $=CuCl_4$.Aq. Indeed, it may be stated that this behavior is the rule, and simple ionization the exception. The fluorides and the cyanides are particularly prone to undergo such ionization when dissolved. On the other hand, salts like the alums and the double sulphates, when treated with water, give solutions in which the simpler ions form the major part of the ions present, although no doubt accompanied by a certain percentage of more complex ions, according to the nature of the salt, the degree of dilution, and the temperature. There can be little doubt, however, that in the solid state, or in the crystalline form, with water of crystallization, it is the complex ions which are present. For instance, a partial formula for potash alum in the crystalline state would be $K-\{Al(SO_4)_2\}.12H_2O$; although in solution the majority of the ions are K.Aq, Al.Aq, $-OH$.Aq, and $=SO_4$.Aq.

It is customary to call salts which possess mainly the latter character "double salts," and those which, like $K_4Fe(CN)_6$, $KAg(CN)_2$, Na_2SiF_6, etc., ionize according to the more complex scheme, "complex salts." But this classification,

although convenient, is not exclusive. It is probable—nay, certain—that Ag ions are present in a solution of potassium argentocyanide because silver can be electrodeposited from its solution. And although it would be impossible to prove that a measurable amount of the silicon ion is present in a solution of sodium silicifluoride, it must be regarded as an extreme case.

Change of valency permits of easy representation on the electronic hypothesis. As an illustration the ferrous and ferric salts may be cited. Supposing the ionization of the chlorides to occur according to the simple scheme, then ferric chloride is $Fe{\equiv}Cl_3^{VII}$, and ferrous chloride, $-Fe{=}Cl_2^{VII}$. The third electron, present in combination with the ferrion in metallic iron, plays no part in the structure of ferrous chloride, but remains latent. It can be brought into action by chlorination, or by oxidation, when the iron "changes its valency." Perhaps the speculation may be here allowed that iron, associated with three electrons, is a less easily attackable body than when one electron is latent. What "latent" in this connection signifies is merely that the latent electron is not so easily transferred as the others. In iron, for example, as in all other elements, the maximum number of electrons associated with an atom appears to be eight. Even when the iron "acts as a triad" there must still be five latent electrons attached to the iron atom—electrons, that is, which play no part in ferric compounds. Some of them, however, are essential when the metal acts as a ferrate, of which more hereafter.

So far, the symbol – (the usual one for a valency or bond) has been employed to denote an electron. This has the convenience of long-established custom; and it also fits in with

the resemblance of the dash to the negative sign, and may be taken also to imply a unit charge of negative electricity associated with the compound or atom. But that sign does not show what direction the electron has taken during the formation of a compound. The idea of direction is easily introduced by the conventional barb of an arrowhead; and $Na \rightarrow Cl$ may signify that during the formation of salt the electron which couples the two atoms was the one which was originally attached to the sodium when it was in the metallic state; and also that on solution in water it will form part of the chlorion.

We have now to consider the electronic formulæ of certain more complex compounds; and as examples two shall be chosen, viz., hydrogen fluoride and ammonium chloride.

In the gaseous state the formula of hydrogen fluoride is H_2F_2; and in solution, $H_2F_2.Aq$. How are the electrons distributed? It is known that hydrofluoric acid may ionize in two fashions: $H.Aq + -HF_2.Aq$ and $2H.Aq + = F_2.Aq$. In the first case the second hydrogen atom does not enter the solution in the ionic state. How is it attached? And how are the two atoms of fluorine combined together? Answers to these questions must necessarily be of a speculative nature; but it appears best to set up a provisional theory which must stand the test of experience and prove compatible with the constitutions assigned to other compounds.

It appears to me that it must be concluded that an atom may have the power both of giving and taking an electron. If a hydrogen atom parts with an electron to a chlorine atom, so that the electron is more closely associated with the chlorine than with the hydrogen atom, then, on solution in water, the hydrion will separate as an entity. If, however, the hydrogen atom H– not merely parts with an electron to an

atom of fluorine, but receives one in return, then: $H \rightleftarrows F$; the hydrogen ion does not separate on addition of water. But by this process the fluorine atom has acquired the property of disposing of an electron which would otherwise remain latent. This serves as the bond of connection between the two fluorine atoms. It may be expressed thus: $H \rightleftarrows F \leftarrow F \leftarrow H$; the kation will then be $H \rightleftarrows F \leftarrow F \leftarrow$, and the anion H.

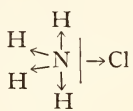
In this manner it may be conceived that a molecule of chlorine may be constituted: $Cl \rightleftarrows Cl$; and its combination with water, preliminary to the attack of sodium, would be thus represented:

$$H_2=O \begin{matrix} \leftarrow Cl \\ \Updownarrow \\ \leftarrow Cl \end{matrix}$$

Let us next consider the case of ammonium chloride. Here we have the group NH_4Cl, which on solution yields the ions $(NH_4).Aq$ and $-Cl.Aq$, similar to common salt. So far the case is clear. How is the group NH_4 to be represented?

The hydrogen of the hydrogen chloride has already given its electron to the chlorine; it has, by hypothesis, no electron to bind it to the nitrogen atom. The effective electron must therefore come from the nitrogen. But there is no known difference between the four hydrogen atoms of ammonium chloride. Of course it is true that when heated one hydrogen atom associates itself with the chlorine (when the vapor is damp); but, so far as is known, any one of the four hydrogen atoms may do so. It may be remarked, in passing, that the fact that two varieties of tetra-substituted ammonium chloride exist has no bearing on the question before us. That has purely stereo-chemical reasons. I suggest as one

solution of the problem that the nitrogen atom parts with its electrons to all four hydrogen atoms, thus:



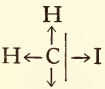
the fifth having become the bond to the chlorine atom. In solution, the group NH_4 is left. The vertical line shows how ionization occurs on solution in water.

This opens the general question, How are the electrons attached in the case of non-ionizable substances, such as PCl_3 or CH_4, to choose only two among almost innumerable instances?

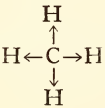
It may be taken as certain that the acidity of an acid is due to the hydrion, H, which accompanies its solution in water; and the residual group may be depicted as $A\leftarrow$. From this it may be argued that, as a rule, where the compound is non-ionizable, not only is the hydrion absent, but the disposition of the electron is such that ionization cannot occur; the hydrogen does not part with its electron to the other element or group. But if the hydrogen were to retain its electron and suffer no further change, it is to be presumed that it would still retain the properties of the element; hence some change must have occurred. Indeed, there are only three possibilities: (1) that the hydrogen parts with an electron, and that has been shown to be practically impossible; (2) that it receives one; and (3) that both its own electron and the one which it receives are the cause of its staying in combination. The first case may be represented, as before, by $A\leftarrow H$; the second, by $X\rightarrow H\rightarrow$; and the third, by $X\rightleftarrows H$.

We know only that methyl iodide, CH_3I, is not generally

regarded as an ionized compound, and yet, on long shaking with a solution of silver nitrate, silver iodide is formed even in aqueous solution. This precipitation takes place more rapidly in alcoholic solution, probably because of the more intimate contact between the reacting bodies. Now the usual representation of such a fact would be $(CH_3)|\rightarrow I$. The group CH_3 has a positive charge; it has lost an electron to the iodine, which has become iodion, $\rightarrow I$. The explanation has already been given in the case of ammonia; it may be symbolized thus:



This leads us to consider the hydrocarbon CH_4; and it may evidently be represented on the same scheme, viz.:



No one of the hydrogen atoms is replaceable; they are all negatively electrified—*i.e.*, to use an exaggeration which will be understood, they are more negatively charged than gaseous hydrogen itself, each atom having received an extra electron. Just as metallic zinc can be preserved from attack by imparting to it a powerful negative charge, so these atoms of hydrogen are rendered inactive by virtue of the protective electrons which they receive from the carbon atom.

Analogous reasoning will prove applicable to compounds like phosphorus chloride; here the electrons possibly come

from the phosphorus atom and from the junctions with the phosphorus atom, thus:

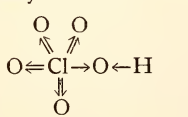
$$Cl \leftarrow P \begin{matrix} \nearrow Cl \\ \searrow Cl \end{matrix}$$

The addition of two atoms of chlorine to form phosphoric chloride would be too speculative to interpret; and it may be remarked that the electrons are probably also derived from the phosphorus.

Let us now return to the halogens, and examine their valency in the light of the electron theory. According to the ordinary view, the acids are, as a rule, dehydrated hydroxides; the elements with high valency do not form normal hydroxides; the known compounds are derived from these hypothetical compounds through loss of water. A single example will render this clear, and the case of chlorine is instructive. The oxy-acids of chlorine are: HClO, hypochlorous acid; $HClO_2$, chlorous acid; $HClO_3$, chloric acid; and $HClO_4$, perchloric acid. In the first of these chlorine functions as a monad; in the second, as a triad; in the third, as a pentad; and in the fourth, as a heptad. The normal hydroxides would be: (1) ClOH; (2) $Cl(OH)_3$; (3) $Cl(OH)_5$; and (4) $Cl(OH)_7$. The first is known as such; the ordinary formula of chlorous acid, as revealed by its salts, is O=Cl–OH; that of chloric acid, $(O_2)^{IV}Cl–OH$; and of perchloric acid, $(O_3)^{VI}Cl–OH$. Now while caustic soda ionizes into hydroxyl and sodium, –OH and Na, hypochlorous acid, which displays some small extent of ionization, in solution gives $(ClO)-$ and H as its ions. And since it is clear in the first case that the atom of sodium metal, in becoming hydroxide, has given an electron to the oxygen, while hydrogen is still retained by the oxygen, the direction of electrons must be $\rightarrow O \leftarrow H$, or possibly $\rightarrow O \rightarrow H-$, or pos-

sibly $\rightarrow O \rightleftarrows H$; in the last case the hydrogen electron as well as that pertaining to the oxygen taking part in the union. Similarly, when hypochlorous acid ionizes, the ions are $(ClO)-$ and H; it is clear that the hydrogen atom has parted with its electron to the group (ClO), and probably to the oxygen atom; we may therefore write its formula $ClO \leftarrow H$. How are the chlorine and oxygen atoms connected?

The highest valency of any element appears to be eight; and in perchlorates that of chlorine is seven. It would be possible for the heptavalent chlorine—*i.e.,* the atom of chlorine stuff combined with seven electrons—to absorb an eighth. That must be supposed to occur in hydrogen chloride; hence we may write its formula $Cl^{VII} \leftarrow H$, the Roman numeral VII expressing the electrons already attached. The formula $Cl^{VII} \leftarrow O \leftarrow H$ would thus portray the condition of the electrons. This, however, gives no clue as to the splitting off of an electronless hydrion in preference to the hydroxyl group $\leftarrow O \leftarrow H$. It may therefore be imagined that one or more of the seven electrons of the chlorine atom take part in retaining the oxygen atom. And this becomes a necessity when we consider one of the higher acids. Were each oxygen atom of perchloric acid to transfer one or two electrons to the chlorine, the latter would be overladen. Hence electrons must be derived from the chlorine; and if each atom of oxygen requires two electrons to bind it (except the hydroxyl oxygen, which, having received one already from the hydrion, will be content with one more), then the formula of perchloric acid may be written:



Chloric acid may be similarly represented; but the chlorine

symbol should be written Cl^{II}, to signify that it still retains two electrons; and the chlorine in chlorous acid retains four out of the seven. These electrons correspond to what have sometimes been called "contra-valencies" (see Abegg's publications).

It is not necessary to multiply instances; the electronic constitution of all the oxy-acids can be thus represented.

We have now a clue to the constitution of the oxides and anhydrides. Orthocarbonic acid, as represented by its esters, will have the formula $C(\rightarrow O\leftarrow H)_4$; carbonic acid, $O\Leftarrow C(\rightarrow O\leftarrow H)_2$; and it follows that CO_2 must be $C(\Rightarrow O)_2$. By analogous formulæ we can represent all the oxides and sulphides of the metals.

The unsaturated oxides call for a short comment. We have CO, also ClO_2, NO and NO_2, as somewhat outstanding compounds. As for CO, it may be treated like the other oxides; or again, it may be that the electrons concerned come from the oxygen, thus: $C^{IV}\Leftarrow O$. The hypothesis, as formerly suggested, that one electron has been derived from the oxygen, the other from the carbon atom, thus: $C^{IV}\rightleftharpoons O$, is not excluded. This would appear, on the whole, the most probable assumption, for in that case the valency of the carbon is not disturbed. On this hypothesis, the formulæ of chlorine peroxide, of nitric oxide, and of nitric peroxide would be $Cl^{VII}\substack{\rightleftharpoons O\\ \rightleftharpoons O}$, $N^{V}\rightleftharpoons O$, and $N^{V}\substack{\rightleftharpoons O\\ \rightleftharpoons O}$, respectively.

We now see that the maximum number of electrons with which an element can be associated is eight. When some of these are employed in holding together the constituent atoms of compounds, the remainder are "latent"; they may under certain circumstances come into action, and in some instances the acting electrons are derived from both constituents of a binary compound. Here we have, perhaps, an explanation of amphoteric bodies—bodies like zinc hydroxide, which can

function either as a base or as an acid. In the former case the electrons would appear to be arranged thus: $Zn(\rightarrow O\rightarrow H)_2$; in the latter, thus: $Zn(\rightarrow O\leftarrow H)_2$. In all probability the oxide is a zincate of zinc, for which an appropriate electronic formula can easily be worked out.

We come next to the consideration of how the electrons form ties between the atoms of compounds; and this must necessarily be of a speculative character. But we have one *point de départ*. We know that a stream of electrons repels a similar stream of electrons passing in the same direction, and attracts a stream of electrons passing in the opposite direction. Reasoning which applies to a number of electrons in all probability applies to single electrons, and it is known to be equally applicable to curvilinear as to straight-line motion. Assuming that the electron connected with an atom of metallic sodium is rotating in a clockwise direction, viewed from the center of the atom, and one of the seven normally attached to an atom of chlorine is rotating in an anti-clockwise direction, combination may be supposed to take place when their planes of rotation become parallel, thus:

Solution in water leaves the sodium atom minus its electron, which remains attached to the chlorine atom.

The applicability of this conception to the position which substituents take up in carbon compounds, and to the influence of groups already present on such position, is easily seen and need not be enlarged on, but they form a not unimportant part of chemistry. The influence exerted on atoms by neighboring atoms and groups also finds ready explanation, and many instances will at once suggest themselves.

Certain compounds show absorption spectra; others do

not, so far as is known, absorb light either in the visible or invisible spectrum. From Zeeman's experiments and Lorentz's theory it is clear that the circular paths usually followed by electrons can be changed to elliptic ones; here we have to some extent a proof that the permanent circular motion of electrons in compounds is not imaginary. · Under the influence of a magnetic field, all compounds which transmit polarized light rotate the plane of polarization; here we have the magnet arranging the planes of rotation of the already rotating electrons, so that they affect ethereal waves passing in their neighborhood.

These rotating electrons represent the "tubes of force" which have sometimes been imagined as the mechanism of chemical attraction. As yet we cannot account for the fact that two electrons, moving in opposed parallel paths, attract each other; it is merely another instance of the inexplicable problem of "action at a distance," which has puzzled all philosophers since the time of Newton and earlier, and even now we are no nearer an explanation. What can be done, however, is to trace the connection between the now known fact that chemical elements and compounds invariably, so far as we know, contain electrons among their constituents, and the mechanism of a compound containing electrons in motion.

In this lecture many of the conceptions are similar to those put forward in the author's presidential address to the Chemical Society of London in March, 1908; hence no allusion has been made here to the possible explanation of the constitution of some complex compounds by the electronic hypothesis; nor has the theory of isorrhopæsis, developed by Baly and his school, been touched on. Reference to the published papers will show how easily their conceptions fit the theory of electrons. It must be distinctly noted that much

of what has preceded is now no longer hypothetical, but actual statement of fact. The electron is no mythical conception, and that it enters into the constitution of matter is as certain as that matter exists.

The combination of one hydrogen atom with another to form a molecule may finally be again considered. Suppose that the constitution of the atom is such that the electron is not free to shift its position on the surface of the molecule, which may be taken as a sphere. If two such atoms have electrons rotating in circular paths on their surfaces in a clockwise direction, then, when the sides of the atoms furnished with electrons are opposite, the electron on one atom will be rotating clockwise, while that on the other will be anti-clockwise, when viewed from the same point. Their orbits being in opposite phase, they will repel. If, however, one is rotating in clockwise and the other in anti-clockwise fashion, as is probably the case with H and Cl, they will attract. This may be depicted thus:

or in abbreviated form, H--Cl.[1]

[1] Note added May 23, 1915: Experiments with models have shown that when two spheres, representing two atoms of hydrogen, each provided with a coil of wire, through which an electric current is passing (so as to imitate the path of an electron), are placed near each other they set themselves (seen from above) in such a position that the two coils of wire lie in the same plane, thus:—

Third Lecture

THE DISRUPTION OF THE SO-CALLED ELEMENTS

THE mechanism by which one element is retained in combination with another has been a matter of frequent speculation. That the properties of atoms depend on their shape was an idea held by the ancients; pointed atoms giving an acid or "sharp" taste to solutions containing them, while the impression of sweetness was imagined to be due to the spherical and smooth nature of the atoms of sugary bodies, and their soothing action on the organs of taste.

After tables of affinity, showing the order of preferential combination of elements with each other, had been drawn up by Bergman of Sweden in the early half of the eighteenth century, the hypothesis was revived that one element is attached to another by means of hooks capable of interlacing. The mechanical nature of this suggestion was at that time hardly a recommendation; and in determining the proportions in which atoms combine, the mechanism of their combination was tacitly ignored.

The prominence given by Frankland and his school to the doctrine of valency, and the important advances in the theory of organic chemistry made by Kekule in the sixties of the nineteenth century, again directed attention to the subject. Although no clear conceptions were formulated, combination was represented by dashes, to which the name of "bonds" or "affinities" was ascribed. Each element capable of combining with or of replacing one atom of hydrogen had attached to its symbol one "bond" or dash, and was termed "monovalent," or a monad; one with the power of retaining

or of replacing two atoms of hydrogen had two "bonds," and was termed a dyad, or divalent; and so with the rest. But no mechanical idea of the nature of these bonds secured acceptance; they were regarded as arbitrary symbols with the signification ascribed to them above.

The fact that less heat is evolved on making attachment of one carbon atom to another by a "double" than by a "single" bond made it improbable that such bonds were of the nature of links or hooks; and the almost thermal neutrality of the "triple" bond strengthened this impression.

Such views, indefinite as they were, tacitly assumed that two atoms when they combine do not interpenetrate; indeed, the notion of an atom as an indivisible entity precluded the conception of interpenetration. But the advance of knowledge, which has rendered it conceivable that atoms may consist of congeries of electrons, now makes it not impossible that the combination of one atom with another may be attended with interpenetration, the one system of electrons entering the other and establishing a more complicated system. Sir J. J. Thomson, some years ago, threw out the ingenious hypothesis that the combination of atoms may be due to the annular rotation of one vortex-ring round another; and he adduced interesting speculations on the number of rings capable of taking part in such annular rotation, and the stability of the resulting system.

But these speculations were anterior to the discovery of the electron as a chemical element; and though by no means to be lost sight of, they may be allowed to stay in the background for the present.

In this lecture I shall attempt to put together evidence which, I hope, will eventually accumulate so as to throw light on the whole question. It will be remembered that in the second lecture the speculation was made that the staying in

combination of two atoms was possibly due to the attraction exercised by their "valency" electrons rotating in similar directions. We have now to consider whether any means can be discovered which will disturb such a system, and by setting free ions in such a condition that they can be examined, light may be thrown on the mechanism of combination; furthermore, whether means are at our disposal of still more fundamentally altering the motion and distribution of the atoms of the "elements," so that a change of the nature of transmutation of one element into another can be effected.

For this end three lines of argument may be adduced. These are:

1. The evidence of the spontaneous disintegration of the radioactive "elements."

2. The evidence that disintegration of a somewhat similar nature occurs in the stars.

3. The evidence that by applying concentrated forms of energy to the common elements, these can be made either to undergo reversible changes, consisting in the loss or gain of one or more "valency" electrons, or to lose more fundamental electrons, and so to undergo "elemental change," or transmutation.

1. The first line of evidence is now so well known that it may be treated in a cursory manner. The discovery by Henri Becquerel that an electroscope is rapidly discharged when in actual communication, by means of a tube, with a vessel in which a salt of radium was contained, was followed by the discovery, by Schmidt, of the transmissibility of a gaseous educt of thorium through a tube. The determination of the nature of this substance by Rutherford and Soddy; the

establishment of the similar nature of the body emitted from salts of radium now known as niton; the proof of its gradual "decay" and reproduction from its parent substance at such rate as to keep the total amount present in contact with the parent body in a state of equilibrium; the naming of these bodies "emanations"; their condensation by liquid air; and lastly, the theory of disintegration applied to such bodies,—all these constitute one of the most brilliant chapters in the history of chemistry.

The subsequent discovery by Ramsay and Soddy, in 1903, of the fact that one of the products emitted during the disintegration of radium and of its emanation was the now well known element helium, and the determination of the gaseous nature of radium emanation by the same investigators; the mapping of its spectrum by Ramsay and Collie, by Cameron, and more accurately by Royds and by Watson; its liquefaction, the measurement of its vapor-pressures, its boiling-point, its critical point, and lastly of its density by Ramsay and Whytlaw-Gray, established the claim of radium emanation to be ranked among the "elements," and to have ascribed to it the systematic name "niton" and the symbol Nt; for its inactivity proves it to belong to the series of inert gases, of which argon is the best known.

It has also been shown by many investigators that from thorium and its educts helium is evolved during their disintegration; and by Debierne, the discoverer of actinium, that it, too, yields helium during its radioactive changes.

Not merely this: the number of atoms of helium evolved during the various changes has been ascertained. The emission of a helium atom takes place with an atomic explosion; the atom evolved has a high velocity—so high that it ionizes any gas through which it passes, and renders the ionized gas capable of discharging an electroscope. It was Becquerel

who first recognized this fact, and who characterized such moving atoms of helium (their nature, however, being at that time unknown) as α-rays, to distinguish them from the even more rapidly moving β-rays, now known to be electrons in motion. Confining our attention for the present to the α-rays, they are known to be evolved when radium disintegrates into niton and helium; for each atom of radium, one α-particle, or helium atom, is evolved. Next, when niton disintegrates, forming radium A, α-particle is again expelled; the spontaneous change of radium A into radium B, however, is accompanied only by the emission of electrons in motion, or β-rays; but the changes of radium B into radium C, and of radium C into radium D, are each accompanied by the emission of an atom of helium. There are in all four atoms of helium expelled between radium itself, the great-great-grandfather, and radium D, the great-great-grandson, and three between niton and radium D. It is not necessary to pass further down the scale, for the life of radium D is a comparatively long one. But three atoms of helium are expelled during the change of niton into radium D, and this fact has been verified by Ramsay and Whytlaw-Gray by aid of the microbalance.

It may be taken as certain, therefore, that radium, as well as thorium and actinium, which undergo analogous changes, is the ancestor of numerous elements. Of the elementary nature of radium and of niton, according to the usual interpretation of the word "element," there can be no doubt whatever; for the former has been isolated in a metallic state by its discoverer, Mme. Curie, and is said closely to resemble barium, an element to which its salts bear a close resemblance; while niton, as previously remarked, bears a close resemblance to the inactive elements of the helium and

argon series, and is a congener of neon, krypton, and xenon, as shown by its physical properties and by its spectrum.

Here, then, we have spontaneous transformation of one elementary form of matter into others. The "elements" are not elementary, but some of them at least are only compounds of exceptional nature, spontaneously capable of decomposition. It remains for us now to determine whether "elements" other than "radioactive elements" are capable of change; and inasmuch as the time required for changes of the kind varies enormously, from millions of years for the change of uranium into radium, of which it appears to be the grandparent, to a few seconds, the half life-period of actinium emanation, it is reasonable to suppose that the periods of change of the older and commoner elements may be enormously long—so long, indeed, as to elude human observation.

But another phenomenon attending the spontaneous change of the radioactive elements must not be left out of sight: the changes alluded to are all accompanied by enormous evolution of heat; they are in the highest degree exothermic; and conversely, if it were possible to produce such radioactive elements by inducing their products of disintegration to combine, an enormous absorption of energy would be essential. Bodies formed by absorption of energy are termed "endothermic"; they are not infrequent among compounds, and they are among the least stable. It is, however, by no means certain that the ordinary elements, which we generally reckon as stable, are endothermic; they may be exothermic, in which case we should expect them to exist indefinitely without change, provided they are not made to receive energy. An analogous case, familiar among compounds, is ammonium chloride. Left to itself, it may be kept for an indefinite time; but when heated to 360° C. it dis-

sociates into ammonia and hydrogen chloride. May it not be the case that the ordinary "stable elements" are in this sense similar to ammonium chloride; that as long as they are not made the recipients of large quantities of energy they remain as they are; but if subjected to an accession of energy, either in the form of heat or of kinetic energy, they may fall apart into simpler forms of matter?

This leads us to the second line of evidence; but before considering it, it may be reiterated that the behavior of the radioactive forms of matter conclusively shows that bodies which have until recently been classed as elementary are in continual process of "disintegration," or, if they be regarded as compounds, of "decomposition."

2. The second line of evidence is dependent on an examination and classification of the spectra of the fixed stars. This work has been accomplished by Sir Norman Lockyer during the past half-century. The arguments for the view that the high temperature of some stars is producing the disintegration of the common elements has been developed by Lockyer in his work entitled "Inorganic Evolution."

In 1864 Mitscherlich showed that certain compounds when heated gave spectra peculiar to themselves and revealing no trace of the elements which they contain. These spectra of compounds present a fluted appearance, the fluting consisting of numerous lines arranged in regular order, and of different intensities. Lockyer showed that at higher temperatures such compounds can be made to exhibit the spectra of the elements which they contain, and the spectra of the elements are characterized by fine lines; the flutings disappear with rise of temperature, to be replaced by the elementary line-spectra.

The effect of rise of temperature on a solid body, such as platinum, is, first, to produce a grayish-white light; this is

succeeded by a dull red, and we say that the solid is "red-hot." A higher temperature increases the amplitude of vibrations, so that the substance emits more light but it diminishes the wave-length, so that the light emitted grows yellower; we may term it "yellow-hot." The next stage is the advent of still shorter vibrations in the green, and when some blue is added the solid is "white-hot," for the combination of these colors produces on our eyes the effect of white light. At a still higher temperature violet light is emitted in quantity, and we might characterize the color seen as "blue-" or "violet-hot."

Taking these colors as a test of the temperature of the stars, Lockyer points out that it is reasonable to group the stars accordingly; and the line spectra of the gases in the stars can then be allocated to a qualitative scale of temperature.

It is possible to imitate similar conditions, although imperfectly, with terrestrial means. The flame is less hot than the arc; the arc gives a lower temperature than the electric spark; and it is possible, by means of powerful discharges, to increase considerably the temperature of the spark. On submitting various ordinary elements to such an ascending scale of temperature, Lockyer noticed that certain spectrum lines became "enhanced,"—*i.e.,* appeared stronger and brighter,—while others diminished in intensity.

On examining the spectra of the stars, it was found that the "enhanced" lines of certain elements were much more prominent in certain stars, and indeed were uncontaminated with the ordinary lines of the elements; these had vanished. Indeed, all stages of change can be followed in the stars; and it is to be noted that as the temperature of the star is higher, the spectra of hydrogen and helium appear, along with spectral lines at present not identified with those of any

terrestrial element. The appearance of these unknown lines is accompanied with the disappearance of the spectrum of the "common" elements, calcium, iron, etc. To the new spectra Lockyer ascribes the coming into existence of new elements, to which he gives the name "proto-elements," regarding them as formed by the disintegration of those known on earth; for instance, he has evidence for the existence of "proto-calcium," "proto-manganese," etc. These "proto-elemental" spectra in still hotter stars are absent, and are replaced by the spectra of helium, oxygen, nitrogen and carbon, along with some unidentified lines; and in still hotter regions, when the spectrum of helium has disappeared, there remains a spectrum whose wave-lengths are related numerically to those of hydrogen, and to this Lockyer has given the name "proto-hydrogen."

If Lockyer's observations and explanations are correct, it would follow that with increase of temperature, matter as we know it undergoes continuous simplification; being finally reduced to one kind, "proto-hydrogen." Whether this final conclusion can be accepted may be left for the present; but as regards his main contention there appears to me to be no manner of doubt.

3. We now pass to the third line of evidence. Let us consider how energy may be applied to "elemental" matter so as to produce disintegration.

First, as regards the nature and amount of energy available two sources present themselves. It is known that both the α-particles—*i.e.*, atoms of helium—and β-particles, or corpuscles—*i.e.*, atoms of electricity or electrons—are emitted from radium and its disintegration-products with enormous velocity. The usual expression for the kinetic energy of a moving body is $\frac{1}{2}\, mv^2$, where m stands for mass and v for velocity. It was shown by Clerk-Maxwell, many years ago,

that the kinetic energy of gaseous molecules could be thus calculated in agreement with experimental results connected with their pressure and temperature. As shown by Joule, kinetic energy can be numerically expressed in heat-units; and the velocity factor of kinetic energy, v^2, corresponds to temperature. This method of presentation has been chosen here because the dissociation of exothermic compound bodies is always a function of temperature; to quote an instance already referred to, ammonium chloride heated to 360° C. is completely resolved into hydrogen chloride and ammonia; and the reverse effect—recombination between these bodies—occurs when the temperature is lowered.

Now the average velocity of an atom or molecule (for in this case they are identical) of helium existing as an ordinary gas at 0° C. is known to be 1.037×10^5 centimeters, or about 1⅓ kilometers, or in English measure approximately half a mile, per second. But the α-particle, or atom of helium, expelled from an exploding radium atom is about 2×10^9 centimeters, or nearly eighty thousand times as great. The squares of these velocities are to each other as the temperatures on the absolute scale; and as 0° C. corresponds to 273 Abs., we have the proportion: $(1.3 \times 10^5)^2 : (2 \times 10^9)^2 :: 273 : 6.5 \times 10^{10}$ degrees Centigrade; the last figure, expressed in words, is the enormous temperature of sixty-five thousand million degrees.

It is difficult to evaluate the temperature of a star; probably even the hottest does not surpass 100,000° C. If that be so, then the effective kinetic energy of an α-particle exceeds that of gaseous atoms in the hottest star by 6.5×10^5, or getting on to a million times. This energy can be brought to bear upon matter by mixing with it a radium salt, or, better, by dissolving in a solution of the compound to be treated some niton; for from niton a much greater number

of helium atoms are expelled in unit time than from even a much greater weight of radium.

The second available source of concentrated energy consists in the utilization of β-rays, either from radium or from the kathode terminal of a high-potential current. No definite experiments have been made with the former, except in so far as it has been shown by Cameron, and subsequently by Usher, working in the laboratory of University College, London, that the available energy of the β-, or kathode, rays from niton does not amount to one-fourteenth of that obtainable from the α-rays, judging by their action in effecting the decomposition of water into hydrogen and oxygen. The effective energy from the kathode terminal of a high-potential current obviously depends on the degree of potential, which is correlated with the velocity of the stream of the electrons, as well as on the quantity of the current, or, in other words, the number of electrons impelled from the kathode. In this latter case it is of course possible to give definite direction to the kathode stream, and even to concentrate it by the use of a spherical or parabolic kathode. Both of these methods have been employed with apparent success.

(1) It has been found on four separate occasions that a solution of copper sulphate, exposed to the action of niton, yields, after removal of the copper and evaporation, a residue in which the spectrum of lithium was recognized. Needless to say, a specimen of the same copper sulphate, under precisely similar conditions except that no niton was added to it, gave no trace of lithium; nor did distilled water containing an equal amount of niton give any mineral residue.

(2) It has also been found that there is a continual evolution of carbon dioxide from a solution of thorium nitrate, left to itself and tested at intervals of six months.

(3) Experiments on the action of niton on a solution of thorium nitrate have resulted in the production of carbon dioxide. The same solution was treated at intervals four times with fresh doses of niton, and the quantity of carbon dioxide formed was roughly proportional to the quantity of niton dissolved in it. The thorium nitrate was finally proved to have contained no compound of carbon.

(4) Other elements of the same group, of which carbon is the member of lowest atomic weight, viz., silicon (as hydrogen silicifluoride), titanium (as titanium sulphate), zirconium (as nitrate), cerium (as sulphate), and lead (as nitrate), were similarly treated with niton; all gave carbon dioxide, with the exception of cerium; and the quantity produced by the same dose of niton was in the same order as the atomic weight of the metal treated; silicon giving least and thorium most. Lead, however, gave a relatively small amount; and from solutions of silver and of mercury (as nitrates) no carbon dioxide appeared to be formed. But from bismuth (as nitrate) a trace was produced.

(5) When water contains niton in solution, it decomposes into oxygen and hydrogen. These gases, formed in relatively large amount, can be removed by explosion, and a constant small excess of hydrogen can be got rid of by addition of pure oxygen and explosion. The excess of oxygen can be withdrawn by exposure to charcoal cooled with liquid air; the residual gas consists of neon mixed with some helium. The presence of neon appears somewhat unaccountable, but it will be seen, further on, that we have a clue to its production; the helium is obviously one of the disintegration products of the niton.

This same change appears to take place in certain mineral springs containing niton; the gases evolved from the hot springs of Bath in England, consisting mostly of nitrogen,

contain about three-quarters as much argon as is normally present in atmospheric air; about sixty-five times as much helium, and about one hundred and eighty times as much neon. Hence, too, the neon evidently arises from the action of dissolved niton on water.

The action of kathode rays on the composition of matter has not as yet been examined, so far as I am aware, except in the two following instances. The first consisted in the examination of the blued glass of bulbs which had been used for the production of X-rays. Four of such bulbs, each of which had served for medical purposes for several months, were broken up, and the fragments of blue glass were placed in a combustion-tube. The air was exhausted and oxygen was admitted, so as to "wash away" all traces of air which might have conveyed with it traces of the gases for which it was prepared to test—helium and neon. The tube was finally exhausted and heated. The gases evolved, mainly oxygen which had been absorbed by the glass, were pumped off, and by the aid of charcoal cooled with liquid air all condensable gases were removed. There was a trace of residual gas, which on spectroscope examination proved to consist mainly of helium; but some feeble neon lines were recognized, showing the presence of a small trace of neon.

Professor Norman Collie undertook the next experiment. It consisted in bombarding with kathode rays a sample of calcium fluoride prepared by the addition of a solution of sodium fluoride to one of calcium chloride. The resulting precipitate was washed, dried and ignited. It was exposed for days to bombardment with kathode rays from a powerful Ruhmkorff coil. But under such circumstances the residual gas in the tube became absorbed, and in order to maintain the vacuum under suitable conditions for a kathode stream, pure oxygen was added from time to time. The first

portions of gas were rejected; but after nearly a week's bombardment about half a cubic centimeter was examined by the method already described. On examination of the gas remaining after absorption of the oxygen, etc., by cooled charcoal, the spectrum of pure neon was noted; helium was absent.

Although it would be inadvisable without further research to dogmatize on the results mentioned, it cannot but be regarded as important that in the absence of oxygen the product should consist of helium, while if oxygen be present neon is formed. The equation $O+He=Ne$, or, in figures, $16+4=20$, would appear to correspond with the change which has occurred. The formation of neon during the action of niton on water would thus also find an explanation, the oxygen being derived from the water and the helium from the niton.

We are merely at the beginning of such work. It will be difficult, owing to the small amounts of matter altered; but methods are being perfected not only to deal with minute quantities of material, but to weigh them with accuracy.[1]

Let us now inquire how electrons may be supposed to take part in such changes. We have as yet no clear mental picture of the structure of an atom; but from what has gone before, it appears evident that certain electrons, in union with the atoms of the substratum of metals, impart to them their metallic nature; it is these electrons which are more or less easily detached, and which correspond to valency. The non-metals appear to be distinguished by the possession of "latent electrons," which come into action during certain conditions of combination, and which also play the part of

[1] Note added May 23, 1915: Several papers have since been published on this subject by Collie and Patterson, by Masson, by Egerton, by Strutt, and by Merton in the "Proceedings of the Royal and Chemical Societies."

valency. The former, attached to metallic substrata, may be exemplified by the metal sodium, which we must now agree to regard as consisting of a substance in union with an electron; the latter, by chlorine, which in the chlorates, perchlorates, etc., develops valencies latent in its monovalent combinations, of which sodium chloride is an example.

Besides these easily detachable electrons, it is legitimate to speculate that whether or no there may be a material substratum to the atom, it contains other electrons, which by their number, their grouping, and their motion play a great part in determining its intrinsic and distinguishing properties.

The collision of an α-particle, or atom of helium, in rapid motion, or of an electron, with an atom, may take place either by a grazing impact or centrally. The chances in favor of a grazing impact are very much greater than those of a central collision. Probably out of every seventy collisions, one is central; the others merely affect what may be termed the "shell" of the atom.

Now it is known that the effect of α-particles or of β-corpuscles on gases is to ionize them. Ionization means the addition of an electron, or of more than one electron, to the atom of a gas; or it may equally mean the removal of one or more electrons from the atom of a gas. In the former case the ion is termed negative; in the latter, positive. Such ions, however, have no permanent existence; given time, they equalize their electric charges, or, in the language of the electronic theory, those having an electron more than necessary for the atomic existence of the gas pass on that electron to those having an electron less. Electric neutrality is thus reëstablished, and the gas loses its conducting power. The action of an α- or a β-particle is, in short, a reversible one, if only the shell of the atom is penetrated. "Valency-electrons"

are added or removed. The colliding α-particles pass through a gas at ordinary pressure for about seven centimeters before the rate of their motion is so diminished that impact with atoms no longer produces an ionizing effect. To use well known and conventional expressions, if the colliding atom or electron becomes slow-moving, then its impact will be so feeble as not to be able to overcome the "affinity" of the ionic electrons for the matter to which they are attached.

In the much rarer cases of a central or nearly central collision, the moving atom of helium or the moving electron penetrates the core of the atoms which it encounters; it must then play great havoc with their structure. The positions and motions of systems of electrons must then be profoundly disturbed; new and stable rearrangements will occur, and other forms of matter will result. In other words, a transmutation will be effected.

It is as certain as any fact can be that the loss of α-particles and of β-corpuscles by radium and its products leads to the transmutation of these bodies into others. They need not belong to the same chemical family; radium itself, a metal of the barium group, by the loss of an atom of helium yields niton, a gas of the inactive series. On the other hand, if reliance can be placed on the results obtained by treatment of members of the carbon column with niton, there is a tendency toward simplification to lower members of the same column; yet cerium, a metal generally regarded as one of the carbon group, fails to yield carbon as the result of disaggregation; and bismuth, an element quite free from any resemblance to those of the niton group, gives some carbon on such treatment.

Again, the formation of neon by the exposure of oxygen and some other body (glass, calcium fluoride?) to the kathode stream opens a way to the synthesis of elements. It

is true that the atomic weight of neon is 20.2, not 20.[1] But the addition of electrons may account for the increase in weight. As some 1800 electrons have a mass equal to that of an atom of hydrogen, the addition to an atom of helium plus one of oxygen of the fifth of 1800, or 360 electrons, would give the necessary increase in atomic weight, and such an addition does not seem impossible.

Nothing has been said regarding the periodic table in what has preceded, except to indicate that transformation does not always take place from the members of any one column to those of lower atomic weight in the same column; and it would at present be premature to speculate as to the ancestry or progeny of the elements. But as a working hypothesis it may be conjectured that while the existence of compounds between what we generally term elements consists in the juxtaposition of their atoms in such a fashion that the electrons of the "valency" order belonging to the combining atoms, which appear to be attached to the surface of the combining atoms, or which at least are easily removed, serve as "bonds" of union, the elements themselves are produced by the interaction of deeper-lying electrons. In fact, when one atom interpenetrates another, so that the deeper-lying electrons of one element influence those of another, what has been termed "transmutation" occurs. Or, conversely, an element of relatively high atomic weight may be induced to split into two or more "elementary" forms of matter; and it would appear probable that in order to produce such a fission the absorption and assimilation of a certain number of electrons is essential; or it may be the loss of some attached electrons. The latter alternative is certainly in operation when radioactive bodies disintegrate.

[1] Note added May 23, 1915: Ashton has brought forward some evidence for the supposition that there are two neons, the atomic weight of one being 20, and of the other, 20.2.

We are still in the dark as regards what happens when a radioactive element undergoes change with no expulsion of an α-particle. A specific instance is the change of radium D into radiums E_1 and E_2, and of the latter into radium F, or polonium. Here there is no helium atom lost; but β-rays, or electrons, are emitted at each stage. These have mass; hence the atomic weight of polonium should be somewhat lower than that of radium D. These two products of the disintegration of niton are at present being investigated at University College, London; and it may be said at once that their reactions are quite distinct, and that they can be separated from each other with the same ease as, say, arsenic can be separated from zinc. Further research will show whether their atomic weights are identical, or whether they differ by a small quantity, as, for example, the atomic weight of nickel differs from that of cobalt.

Attempts have been made, although with no definite results, to determine whether an "allotropic change"—*e.g.*, that of ozone into oxygen, or that of red into yellow phosphorus—is attended by the gain or expulsion of electrons. But it must be remembered that the usual test for electrons depends on the ionization of air by rapidly moving electrons, and that it is difficult to recognize electrons unless they are in rapid motion. It is true that an electric charge can be tested for and measured; but the existence of an electric charge is no proof that what may be termed "elemental electrons" have been gained or lost; the charge may be due to the gain or loss or the transference of "valency-electrons." It has thus not been shown that allotropy is due to gain or loss of elemental electrons; in all probability it is not, but to the familiar rearrangement of the atoms of a compound, for which we generally use the term "isomerism."

Enough has now been said to show the nature of the prob-

lems which await solution. Progress must of necessity be slow; but methods of micro-analysis have now been much improved, and the microbalance affords a means whereby quantities of matter of the small order which must be handled can readily be weighed. The field is ripe unto the harvest, but as yet the laborers are few.

WILLIAM RAMSAY.

Carl Störmer

THE CORPUSCULAR THEORY OF AURORA BOREALIS *

IN the following pages I shall have the honor to give a résumé of my researches on aurora borealis, begun in the year 1904 and continued up to the present time. Most of the results have been published in "Videnskabsselskabets Skrifter," † Christiania, and in the "Archives des Sciences Physiques et Naturelles," ‡ Geneva.

1. *Introduction.*

It seems to be Goldstein who has the priority in the idea that the sun sends out into space electrical rays analogous to cathode-rays, and that this may explain the mysterious connection between variations in solar activity and corresponding fluctuations in the magnetic and electric phenomena on the earth. This idea was published in "Wiedemann's Annalen" in 1881, in a paper entitled "Ueber die Entladung der Electricität in verdünnten Gasen." §

Some time later, the Danish meteorologist, Adam Paulsen, was led, from his observations of aurora in Greenland, to the

* A lecture presented at the inauguration of the Rice Institute, by Carl Störmer, Professor of Pure Mathematics at the University of Christiania.

† "Sur le mouvement d'un point matériel portant une charge d'électricité sous l'action d'un aimant élémentaire," *l.c.*, 1904, and "Bericht über eine Expedition nach Bossekop zwecks photographischer Aufnahmen und Höhenmessungen von Nordlichtern," *ibid.*, 1911.

‡ "Sur les trajectoires des corpuscules électrisés dans l'espace sous l'action du magnétisme terrestre, avec application aux aurores boréales," etc., *l.c.*, 1907. *Ibid.*, second mémoire, *l.c.*, 1911 and 1912. In the second memoir I have given a list of the twenty-four papers that I have published on auroras up to the year 1912.

§ "Wiedemann's Annalen," Vol. XII, 1881, p. 266.

hypothesis * that aurora was due to cathode-rays; but instead of assuming that these rays came from the sun, he thought they had their origin in the upper atmosphere.

Then, in 1896, came Professor Kr. Birkeland's experiments on what he called "the suction † of cathode-rays towards a magnetic pole." He found that a magnetic pole had an effect upon a beam of parallel cathode-rays analogous to that of a lens upon a beam of light, namely, to make them converge toward a point. ‡ This phenomenon led him, independently of Goldstein, to the idea that aurora was due to a similar effect of the earth's magnetism on cathode-rays coming from the sun, especially from the sun-spots.

To test this hypothesis, Professor Birkeland exposed a little spherical electromagnet to a stream of cathode-rays, and found a series of analogies to the shape and nature of the aurora. The auroral belts in particular were very beautifully produced. These remarkable experiments, which gave the first really good support to the corpuscular theory of aurora, were described in the paper on his aurora expedition of 1899–1900; § but the photographs were not published until 1907.‖

Figs. 1 and 2 show these artificial auroral belts round the polar regions of the little magnetic sphere in Professor Birkeland's experiments.

Notwithstanding that these remarkable experiments tend to show that aurora is a *direct* effect of the precipitation of

* Adam Paulsen, "Sur la nature et l'origine de l'aurore boréale," Copenhagen, 1894.

† His expression is very badly chosen; in fact, the magnetism has no attraction on cathode corpuscles, but only a deviating action, as is well known.

‡ "Archives des Sciences Physiques et Naturelles," Geneva, 4 période, Vol. I, p. 497.

§ "Videnskabsselskabets Skrifter," Christiania, 1901.

‖ Professor Birkeland allowed me to publish some of these photographs in my paper, "Sur les trajectoires des corpuscules," etc., *l.c.*

cathode-rays in the upper atmosphere, Professor Birkeland considered aurora more as a *secondary phenomenon* * due to secondary rays from great electric currents in the upper air; but he also admitted the existence of a direct action.†

In the mean time Arrhenius ‡ published his hypothesis that the sun was sending out small electrified particles from about one ten-thousandth to one thousandth of a millimeter in diameter, and that these particles were pushed away from the sun by the pressure of the light, and on reaching the earth's atmosphere caused aurora.

In the beginning of the year 1903 I was becoming extremely interested in Professor Birkeland's experiments and theory of aurora, and knowing that the phenomenon of the concentration of cathode-rays toward a single pole had been mathematically treated by Poincaré,§ I thought it might be interesting to find out mathematically the trajectories of electrified corpuscles in the magnetic field of the earth, and hoped in this way to find again, not only the details of Professor Birkeland's experiments, but also the principal features of aurora and of the magnetic storms. My first results were then published in "Videnskabsselskabets Skrifter" in 1904. ‖

The work has been since continued, and in 1907 the first detailed report was published in "Archives des Sciences Physiques et Naturelles," Geneva.

In the following paragraphs we will give a short account of those results.

* "Expedition Norvégienne de 1899–1900 pour l'étude des aurores boréales," pp. 60–74.

† *Ibid.*, p. 74.

‡ "Ofversigt af Kongl. Vetenskaps-Akademiens Förhandlingar," 1900; Stockholm.

§ "Comptes Rendus," Paris, Vol. CXXIII, p. 930, 1896.

‖ "Sur le mouvement d'un point," etc., *l.c.*

2. *Simplifying hypothesis for the mathematical treatment.*

Starting with the hypothesis that the sun is sending out electrical corpuscles towards the earth, the mathematical problem to find the trajectories of those corpuscles is an extremely difficult one to solve in its most general form.

As I pointed out in my Geneva paper, the natural way to proceed should be this: first, to try to solve the problem in a series of simplifying hypotheses, and after that to treat the cases in which these simplifying hypotheses are abandoned, one after another, in order to get the real conditions of nature.

As simplifying hypotheses I chose, in my Geneva paper, the following:

I. The motions of the earth and of the sun are considered as negligibles, so that only their relative positions come into consideration; in fact, the speed of the electrified corpuscles considered is supposed to be so great that this relative position does not sensibly change during the time a corpuscle takes to go from the sun to the earth.

II. We assume that the corpuscles are not affected by other forces than the earth's magnetism, and

III. That they follow the laws observed for the motion of a cathode-particle in a stationary magnetic field.

IV. As regards the earth's magnetism, we consider it, in accordance with Gauss's hypothesis, as due exclusively to magnetic masses in the interior of the earth, so that we have the known expansion of the magnetic potential outside the earth in a series of spherical harmonics.

V. In the mathematical analysis we use only the first term of the series for the potential, which means that we consider the earth's magnetic field as a field due to an

elementary magnet placed in the center of the earth with its axis coinciding with the magnetic axis of the latter.

Under the above-mentioned hypothesis, this approximation will hold good at great distances from the earth, because the other terms of the potential expansion containing higher powers of $\frac{R}{r}$ will be negligible as compared with the first term. (Here R is the radius of earth, and r the radius vector.)

The problem is thus reduced to a study of the trajectories of electrified corpuscles in the field of an elementary magnet. When this problem has been solved, we may successively have to take into account the following more general problems:

As regards hypothesis I: To take into account the motion of the earth and of the sun during the motion of the corpuscles.

As regards hypothesis II: To take into account the possible electromagnetic fields surrounding the celestial bodies, especially the sun. Further, other forces that may act on the corpuscles, such as gravitation and the pressure of light in Arrhenius's theory.

As regards hypotheses III and IV: To take into account the reciprocal electromagnetic action between the corpuscles when their number is considerable. Each corpuscle carries with it an electromagnetic field, and the changes in these fields are transmitted through space with the velocity of light. Especially, to take into account the magnetic field outside the earth produced by currents of corpuscles when their number is considerable, as probably during magnetic storms.

As regards hypothesis V: To take into account the real Gaussian expression for the magnetic field with all the terms hitherto considered.

There are, as will be seen, enough difficult problems to solve even when the first one, corresponding to hypotheses I–V, is completely cleared up. It is a promising circumstance that the solution of this first problem in itself gives a good explanation of a series of the principal features of the phenomena of auroras and magnetic storms.

I

MATHEMATICAL DISCUSSION OF THE MOTION OF A CORPUSCLE IN THE FIELD OF AN ELEMENTARY MAGNET

3. *Differential equations of the trajectory in the case of an elementary magnet.*

We will now study the trajectories of electrified corpuscles in the field of an elementary magnet.

We take as unit a length c centimeters, where c is given by the relation

$$c = \sqrt{\frac{M}{a}}.$$

Here M is the moment of the elementary magnet, and a is a constant characteristic of the corpuscle in motion. Let us, for instance, consider a point in the trajectory where the tangent is at right angles to the magnetic force; let us denote with ρ the radius of curvature in centimeters at that point, and presume that the magnetic force is equal to H magnetic units; then

$$a = H\rho.$$

If we put the elementary magnet at the origin of a rectangular Cartesian system of coördinates $OXYZ$ (see Fig. 3), with its axis coinciding with the Z-axis, and its south pole towards the positive z, then the components of the magnetic force * at a point (x, y, z) will, by definition, be the partial derivatives of the function

$$M\frac{z}{r^3}.$$

* That is, the force acting on a unit of north magnetism.

With the adopted unit of length, the differential equations of the trajectory * for a negatively charged corpuscle will be

$$\left.\begin{aligned} r^5\frac{d^2x}{ds^2} &= 3\,yz\frac{dz}{ds} - (3\,z^2 - r^2)\frac{dy}{ds} \\ r^5\frac{d^2y}{ds^2} &= (3\,z^2 - r^2)\frac{dx}{ds} - 3\,xz\frac{dz}{ds} \\ r^5\frac{d^2z}{ds^2} &= 3\,xz\frac{dy}{ds} - 3\,yz\frac{dx}{ds} \end{aligned}\right\} \qquad (1)$$

where $r^2 = x^2 + y^2 + z^2$ and where we have taken as independent variable the arc s of the trajectory.

For a positively charged corpuscle, the signs of the second members of the equations have to be reversed; but the same effect is obtained by changing x into $-x$, that is, by changing the positive direction of the X-axis. *Hence the trajectories of positive corpuscles will, for the same value of $H\rho$, be symmetrical with the trajectories of negative corpuscles relatively to a plane through the Z-axis.* It is of course sufficient to study the latter trajectories.

By introducing polar coördinates R and ϕ, defined by the equations (see Fig. 4)

$$x = R\cos\phi, \quad y = R\sin\phi,$$

we obtain from the two first equations

$$\frac{d}{ds}\left(R^2\frac{d\phi}{ds}\right) = -\frac{3\,R^2z}{r^5}\frac{dz}{ds} - \frac{r^2 - 3\,z^2}{r^5}R\frac{dR}{ds},$$

where the second member is the exact derivative of the function R^2r^{-3}; in integrating we thus obtain

$$R^2\frac{d\phi}{ds} = 2\gamma + \frac{R^2}{r^3}, \qquad (2)$$

* See my Geneva paper of 1907.

where γ is a constant of integration. If we eliminate ϕ by means of this equation, and call Q the following function of R and z,

$$1-\left[\frac{2\gamma}{R}+\frac{R}{r^3}\right]^2$$

we get the very simple system for R and z as functions of s:

$$\left.\begin{aligned}\frac{d^2R}{ds^2}&=\frac{1}{2}\frac{\delta Q}{\delta R},\\ \frac{d^2z}{ds^2}&=\frac{1}{2}\frac{\delta Q}{\delta z},\\ \left(\frac{dR}{ds}\right)^2+\left(\frac{dz}{ds}\right)^2&=Q.\end{aligned}\right\}\qquad(3)$$

The problem is thus reduced to the integration of this system, which can be done by integrating a differential equation of the second order followed by a quadrature. After that ϕ will be found by a new quadrature.

But even without integrating the differential equations it is possible, as will be shown, to draw very interesting conclusions directly from the equations, and further, by the powerful methods of numerical integration, to calculate trajectories with any accuracy desired.

4. *Formula for sin θ. Part of space beyond which the trajectory cannot go.*

From formula (2) we obtain a very interesting geometrical property of the trajectory; if we call the angle between the tangent in the direction of motion and a plane passing through the point of contact and the Z-axis θ, then

$$\sin\theta=\frac{Rd\phi}{ds},$$

and equation (2) then gives

$$\sin\theta = \frac{2\gamma}{R} + \frac{R}{r^3}. \tag{4}$$

By this equation the angle θ can be found for each point of the trajectory; and θ will be positive or negative according as the motion along the trajectory has the same direction as the positive direction of the angle ϕ or the opposite. (See Fig. 5.)

From formula (4) many interesting conclusions may be drawn. For instance, in the region where $2\gamma R^{-1} + Rr^{-3}$ is positive, θ will be positive and the angle ϕ will be constantly increasing; and where the expression is negative, ϕ will decrease.

On the other hand, along a trajectory ϕ can only reach its maximum or its minimum in the points of intersection with the surface

$$\frac{2\gamma}{R} + \frac{R}{r^3} = 0.$$

This surface, which only exists for negative values of the constant γ, is a surface of revolution obtained by rotating a line of magnetic force around the Z-axis.

For a given trajectory, the corresponding value of the constant γ can be found by equation (4) by substituting for a given point the values of R, r and $\sin\theta$, which immediately give γ.

But the most valuable consequence of relation 4 is the following:

Along a trajectory, $\sin\theta$ cannot be less than -1, nor greater than $+1$; the trajectory must of course be confined to the region in space where

$$-1 \lessgtr \frac{2\gamma}{R} + \frac{R}{r^3} \lessgtr +1.$$

We will call this region $Q\gamma$. To each value of γ we obtain a corresponding $Q\gamma$ and no trajectory corresponding to the same value of the constant γ can get beyond this region $Q\gamma$.

To find $Q\gamma$ we may proceed in the following way: Put $\sin\theta = k$ and $R = r\cos\psi$; then the intersection between the surface of revolution where $\sin\theta = k$ and a plane through the Z-axis will be a curve whose equation in polar coördinates found by (4) is

$$kr^2\cos\psi - 2\gamma r - \cos^2\psi = 0. \qquad (5)$$

If we then let k vary between -1 and $+1$, this curve describes in the above-mentioned plane a region which we will call q_γ. In rotating this region around the Z-axis, we then obtain the region $Q\gamma$.

The detailed discussion of the curves (5) will be found in my Geneva paper, as also the discussion of the parts $Q\gamma$ for all values of γ between $-\infty$ and $+\infty$.

We will here only give six characteristic forms of q_γ and $Q\gamma$ corresponding to

$$\gamma = -1.016$$
$$\gamma = -0.97$$
$$\gamma = -0.5$$
$$\gamma = -0.05$$
$$\gamma = 0.03$$
$$\gamma = 0.2$$

In the upper row of Plate I we see the shapes of the regions q_γ white, the other parts of the plane being black. The origin is in the middle, and the dotted lines are the lines of magnetic force, where $\sin\theta = 0$. The unit of length $\sqrt{\frac{M}{H\rho}}$ is equal to the largest diameter of the dotted oval corresponding to $\gamma = -0.5$.

In the lower row are seen the corresponding regions $Q\gamma$ in space, described by the parts q_γ when rotated about the Z-axis.

We find especially that the regions $Q\gamma$ is open from the origin to infinite distance only if

$$-1 < \gamma \leq 0.$$

This, as we shall see, will have important consequences in its application to aurora.

5. *Mechanical interpretation of system* (3) *and results for the discussion of the trajectories.*

Still more useful information concerning the trajectories is obtained if we interpret system (3) mechanically; in fact, if we consider s as the time and R and z as the Cartesian coördinates of a material point p in a plane, then system (3) defines the motion of that point under the action of a force derived from the function of force $\frac{1}{2}Q$. As such a plane we may choose an arbitrary fixed plane ORZ through the Z-axis. Further, let P be a moving point on the trajectory, and let us lay a circle through P parallel to the XY-plane, and with its center on the Z-axis. Then p will be the point of intersection between this circle and the plane ORZ; and when the point P is moving with constant velocity along the trajectory T, the corresponding point p is moving in the plane ORZ according to the above-mentioned mechanical law, and will describe a certain plane curve K.

When we know the shape of the curve K, the shape of the corresponding trajectory in space is easy to find by the formula for $\sin\theta$.

To each curve K there are in general two corresponding sets of trajectories, each containing all trajectories that can be obtained from one of them by rotation around the Z-axis;

the first set corresponds to a motion along K in one direction, the second to a motion in the opposite direction, and the two sets are symmetrical with one another with reference to the RZ-plane.

Now the study of the curves K is comparatively easy when the level-lines

$$Q = h$$

are drawn for a series of equidistant values of the constant h. These lines are identical with the lines (5), where $k = \pm\sqrt{1 - h}$, as we see by substituting the value of the function Q. Of course these level-lines are situated exclusively in the plane region before called q_γ, and the boundaries of that region are formed by the level-lines

$$Q = 0.$$

The line of force $k = 0$ is identical with the level-line $Q = 1$.

I have drawn such level-lines corresponding to equidistant values of h with interval 0.1 for a series of characteristic values of the constant of integration γ, and in order to facilitate the mechanical interpretation, have colored the parts between successive level-lines with graduated tints — white nearest the lines $Q = 1$, and dark nearest the line $Q = 0$. On Plates II to XIX are seen the fields of force thus constructed for γ equal to -1.2, -1.016, -1.001, -0.999, -0.97, -0.9, -0.8, -0.7, -0.6, -0.5, -0.4, -0.3, -0.2, -0.1, -0.01, 0, 0.01, and 0.1 respectively; on Plate XX is also seen the inner part of the field of force for $\gamma = -0.5$ on a scale ten times as great.

The force acting on p will then always be directed normally to the level-lines and toward the lighter parts, and the strength of the force will be approximately inversely proportional to the breadth of the spaces between two consecutive level-lines.

A very intuitive idea is already obtained about the curves K if we consider the point p as a small sphere rolling without friction in a landscape where the level-lines indicate the shape as on geographical charts, the valleys being light and the higher parts darker. The analogy is not complete, but gives nevertheless a fair idea of the form of the curves K.

6. *The methods of graphical and numerical integration applied to the study of the curves K and the trajectories in space.*

The above-mentioned methods are excellent for the qualitative discussion of the trajectories. For the quantitative investigation, however, they are not sufficient, and it is then necessary to use methods of graphical and numerical integration, the first when no very great accuracy is required, the second in those cases in which the greatest possible accuracy is necessary.

The method of graphical integration that I have employed is described in a paper* published in 1908. It is based on a further development of an idea of Lord Kelvin's, and supposes the level-lines $Q = h$ to be drawn; the radius of curvature can then be approximately found by a very simple construction.

The method of numerical integration is by far the more accurate. It has been described in detail in my Geneva paper of 1907, and is analogous to the methods used in astronomy for calculating orbits. In the first place an integral curve K of system (3) has been computed and then the corresponding trajectory in space has been found by numerical quadrature. The computation has been made throughout to six places of decimals, and has been most

* On the graphic solution of dynamical problems. ("Videnskabsselskabets Skrifter," Math. naturv. Kl. 1908; Christiania.)

tedious and elaborate. A computer with enough practice can calculate only about 3 points of a trajectory in an hour.

My assistants and myself have used this method of numerical integration for some years, and have computed more than 120 different branches of trajectories, a labor of more than 5000 hours. But these computations have been of the greatest importance in the applications, and at the same time a most interesting test of the theoretical development, and have given very suggestive ideas.

The detailed computations have not yet been published,* only figures and wire models of the trajectories.

7. *First general view of the trajectories corresponding to a wire model constructed by graphical integration.*

Space does not permit us to give here a detailed description of the trajectories; we can only point out some general characteristics.

We will begin by showing a picture of a wire model constructed by graphical integration, published † in my lecture held at the International Congress of Mathematics in Rome, in 1908.

Here the elementary magnet is placed in the center of the sphere, and the Z-axis is normal to the plane of the model, *i.e.* parallel to the dark supporting rods. The white wires represent trajectories of corpuscles coming from the square plate at the right. Our unit of length is equal to the radius of the circular trajectory round the sphere, and the XY-plane is the plane of that circle. In the XY-plane are also represented the particular trajectories lying in that plane,

* The computations have meanwhile been published in "Videnskabsselskabets Skrifter," 1913 and 1914, Christiania. (Remark during the correction of proofs.)

† See my Geneva paper of 1907, § 20.

trajectories calculated by elliptic integrals, *i.e.* exactly;* they are a good check on the others, which are found by graphical integration.

Only trajectories coming from points in the plate above or in the XY-plane are seen in the wire model. To the former of these there are corresponding trajectories coming from points below the XY-plane and symmetrical with them with reference to that plane.

In Fig. 7 is seen the same wire model with the square plate in the background. The shape of the trajectories is easily understood when compared with the corresponding integral curves K in the RZ-plane:

Those farthest to the right in Fig. 7, and along which the angle θ is positive, correspond to the case in which the constant γ is positive; the corresponding curves K run upward as seen in Fig. 8.

The next trajectories above the little sphere correspond to γ about -0.3. Along these θ is at first positive and then negative, and the angle ϕ goes through a maximum where the corresponding K-curves intersect the lines of magnetic force (see Fig. 9).

Those right up to the left of the little sphere are seen forming a whirl, and come nearer to the sphere than the others. They correspond to γ nearly -0.5 and to integral curves K going into the horn of the region q_γ (see Fig. 10).

The following trajectories, some of which are seen going down under the XY-plane and bending upward again to the right of the sphere, belong to an extremely interesting family studied in detail in a paper published in 1911.† They

* See also Professor Kr. Birkeland's work, "The Norwegian Aurora Polaris Expedition," 1902–03, Vol. I, first section, p. 156.

† "Sur une classe de trajectoires remarquables dans le mouvement d'un corpuscule électrique dans le champ d'un aimant élémentaire." ("Archiv for mathematik og naturvidenskab," Vol. XXXI, No. 11; Christiania, 1911.)

correspond to integral curves K penetrating through the "défilé" between the two parts of the field of force q_γ for γ between -1 and -0.8 (see Fig. 11).

Farthest to the left are seen the trajectories corresponding to γ negative and greater than -1 in numerical value. Along these trajectories the angle θ is always negative and ϕ always decreasing. They correspond to curves K in the outer part of the region q_γ as seen in Fig. 12. They all turn their concavity toward the sphere.

We see that these two last-mentioned groups of trajectories form a thick bundle of curves that more or less encircle the little sphere on the afternoon and night side, if we consider the sphere to be the earth and the square plate the sun sending out the corpuscles.

Most of the trajectories, moreover, only approach the sphere to within a certain distance, and then go out again into infinity. Only those in the middle of the whirl for γ about -0.5, and those corresponding to curves K through the "défilé" for γ between -0.8 and -1 contain trajectories that can reach theoretically the origin of coördinates.

Those trajectories which pass through the origin and extend to an infinite distance are now of the most fundamental importance in their application to the aurora borealis. They have received special study, as we shall see in the following paragraph.

8. *The trajectories passing through the origin, and their computation.*

For each negative value of γ there are in general two curves K passing through the origin. These curves are lying symmetrically with the R-axis. To each curve there are two corresponding trajectories in space, T_1 and T_2, T_1 for a

motion toward the origin and T_2 for a motion from it. T_1 and T_2 are symmetrical with reference to a plane through the Z-axis. Then, as we have seen in § 5, there is, corresponding to T_1, an infinite number of trajectories obtained by rotating T_1 about the Z-axis; they are all congruent. In the same way there is, corresponding to T_2, an infinite number of trajectories obtained by rotating T_2 about the same axis; they are also all congruent.

The study and calculation of these trajectories by the method of numerical integration have been carried out in great detail in my Geneva paper of 1907. The inner parts of the computed curves K are seen on Plate XXI, and many of the corresponding trajectories in space in the wire model in Fig. 13. The entire computation has taken more than 700 hours, and has been made to six places of decimals. As an example are here given the coördinates R, z and ϕ corresponding to values of the arc s for one of these trajectories.

The arc s is reckoned from the origin, and s_0 is the value corresponding to the starting-point. Here $s_0 = 0.2368$. ϕ is zero for $s = 0$.

Trajectory through the origin corresponding to $\gamma = -0.8$

$s - s_0$	R	z	$\phi°$	$s - s_0$	R	z	$\phi°$
0	0.139305	0.182864	0.783	13	0.183969	0.206944	1.220
1 : 256	0.142628	0.184916	0.813	14	0.187523	0.208563	1.259
2	0.145971	0.186935	0.843	15	0.191092	0.210148	1.300
3	0.149334	0.188920	0.874				
4	0.152717	0.190871	0.906	8 : 128	0.194676	0.211699	1.340
5	0.156120	0.192788	0.939	9	0.201886	0.214700	1.422
6	0.159542	0.194672	0.971	10	0.209150	0.217565	1.508
7	0.162982	0.196524	1.004	11	0.216463	0.220294	1.597
8	0.166439	0.198343	1.038	12	0.223824	0.222884	1.689
9	0.169913	0.200129	1.073	13	0.231230	0.225340	1.784
10	0.173403	0.201883	1.108	14	0.238678	0.227661	1.882
11	0.176909	0.203604	1.144	15	0.246165	0.229848	1.981
12	0.180431	0.205297	1.181	16	0.253688	0.231902	2.084

Trajectory through the origin corresponding to $\gamma = -0.8$ — *Continued*

$s - s_0$	R	z	$\phi°$	$s - s_0$	R	z	$\phi°$
17	0.261244	0.233823	2.190	23	1.224723	− 0.271145	41.572
18	0.268832	0.235611	2.300	24	1.260771	− 0.298668	43.554
19	0.276448	0.237267	2.412	25	1.297250	− 0.325358	45.487
20	0.284089	0.238793	2.526				
21	0.291753	0.240189	2.644	13 : 8	1.334235	− 0.351250	47.369
				14	1.409961	− 0.400800	50.969
11 : 64	0.299437	0.241455	2.766	15	1.488260	− 0.447662	54.347
12	0.314855	0.243598	3.018	16	1.569262	− 0.492185	57.502
13	0.330323	0.245230	3.283	17	1.652971	− 0.534700	60.436
14	0.345821	0.246359	3.560	18	1.739311	− 0.575503	63.161
15	0.361331	0.246995	3.848	19	1.828153	− 0.614856	65.686
16	0.376835	0.247148	4.148	20	1.919345	− 0.652981	68.026
17	0.392316	0.246829	4.459	21	2.012720	− 0.690069	70.193
18	0.407758	0.246050	4.781	22	2.108110	− 0.726280	72.201
19	0.423146	0.244824	5.114	23	2.20535	− 0.761748	74.062
20	0.438466	0.243165	5.456	24	2.30429	− 0.796586	75.789
21	0.453705	0.241086	5.808	25	2.40478	− 0.830888	77.394
22	0.468851	0.238602	6.170				
23	0.483893	0.235727	6.542	13 : 4	2.50669	− 0.864733	78.886
				14	2.71428	− 0.931304	81.573
12 : 32	0.498821	0.232476	6.923	15	2.92621	− 0.996709	83.917
13	0.528298	0.224904	7.711	16	3.14179	− 1.06124	85.973
14	0.557218	0.216006	8.533	17	3.36042	− 1.12511	87.788
15	0.585534	0.205902	9.386	18	3.58161	− 1.18846	89.399
16	0.613208	0.194710	10.268	19	3.80513	− 1.25142	90.837
17	0.640217	0.182545	11.177	20	4.03048	− 1.31406	92.127
18	0.666548	0.169518	12.110	21	4.25748	− 1.37645	93.289
19	0.692198	0.155734	13.065	22	4.48590	− 1.43864	94.341
20	0.717171	0.141293	14.041	23	4.71556	− 1.50066	95.298
21	0.741481	0.126291	15.035				
22	0.765146	0.110815	16.046	12 : 2	4.94630	− 1.56255	96.173
23	0.788190	0.094947	17.072	13	5.41055	− 1.68601	97.707
24	0.810641	0.078763	18.112	14	5.87782	− 1.80915	99.008
25	0.832530	0.062331	19.163	15	6.34750	− 1.93206	100.127
				16	6.81913	− 2.05480	101.098
13 : 16	0.853891	0.045715	20.224	17	7.29237	− 2.17742	101.946
14	0.895176	0.012152	22.370	18	7.76693	− 2.29994	102.697
15	0.934796	− 0.021538	24.539	19	8.24261	− 2.42239	103.363
16	0.973054	− 0.055045	26.719	20	8.71923	− 2.54478	103.960
17	1.010243	− 0.088135	28.901	21	9.19666	− 2.66712	104.496
18	1.046642	− 0.120630	31.076				
19	1.082504	− 0.152407	33.235	11	9.67478	− 2.78943	104.981
20	1.118056	− 0.183383	35.370	12	10.63276	− 3.03395	105.823
21	1.153499	− 0.213509	37.476	13	11.59259	− 3.27839	106.529
22	1.189006	− 0.242763	39.544	14	12.55386	− 3.52277	107.129

Trajectory through the origin corresponding to $\gamma = -0.08$ — *Continued*

$s - s_0$	R	z	$\phi°$	$s - s_0$	R	z	$\phi°$
15	13.51628	− 3.76710	107.646	40	37.6957	− 9.87207	112.023
16	14.47962	− 4.01140	108.096	44	41.5710	− 10.8487	112.253
17	15.44372	− 4.25568	108.491	48	45.4469	− 11.8254	112.445
				52	49.3232	− 12.8021	112.606
18	16.40845	− 4.49994	108.840	56	53.1998	− 13.7787	112.744
20	18.33945	− 4.98841	109.430	60	57.0767	− 14.7554	112.864
22	20.27200	− 5.47684	109.909	64	60.9538	− 15.7320	112.968
24	22.20570	− 5.96524	110.305	68	64.8310	− 16.7087	113.060
26	24.14028	− 6.45362	110.639				
28	26.07555	− 6.94199	110.924	72	68.7085	− 17.6853	113.141
30	28.01137	− 7.43035	111.170	80	76.4637	− 19.6386	113.280
32	29.9476	− 7.91870	111.384	88	84.2194	− 21.5919	113.393
34	31.8842	− 8.40705	111.573	96	91.9754	− 23.5452	113.487
36	33.8212	− 8.89539	111.739	104	99.7315	− 25.4985	113.566
38	35.7583	− 9.38373	111.888	112	107.4880	− 27.4518	

The calculated trajectories through the origin are only the simplest ones, as I pointed out in my Geneva paper. For γ between -0.93 and -1 there is an immense number of remarkable trajectories not yet studied in detail, corresponding to curious K-curves passing through the défilé from the outer to the inner region of the field of force q_γ. There are, for instance, as already pointed out in my Geneva paper, trajectories that extend round the Z-axis in waves up and down through the XY-plane, and whose number of revolutions round the Z-axis may be greater than any number given beforehand; they come very near to certain periodic orbits in the neighborhood of the circular orbit in the XY-plane.

From the calculated trajectories it is possible by interpolation to find any trajectory corresponding to γ between zero and -0.93, and thus, for instance, to construct the trajectories going out from a given point, and reaching the origin, for that interval of γ.

On Plate XXII are seen those trajectories for different positions of the point of departure. The Z-coördinates are written beside the marked points of the trajectories, so that any one can make a wire model of the figures if he likes. The trajectories going to the upper part of the sphere round the origin are plain; those going to the lower part are dotted. The trajectories are drawn only until their intersection with the sphere. A wire model is seen in Fig. 14.

The figures illustrate the important theorem that through a given point in space there are generally a series of *distinct* trajectories passing through the origin.

As developed in my Geneva paper, the number of such distinct trajectories from a given point to the origin may vary immensely; it may happen that we have a series of an infinite number of trajectories corresponding to an infinite number of values γ converging to a limit γ situated between -0.93 and -1. There may even be several series like this, and probably even an infinite number of such series of trajectories.

9. *The periodic trajectories.*

It is very interesting that there exists an infinite number of trajectories composed of identical parts, so that a corpuscle following such a trajectory will have a periodic motion. There are even closed trajectories of this kind, so that the corpuscle, after a certain time, comes back to the same point with the same direction of velocity as before.

The problem of finding these periodic orbits is very much facilitated by the mechanical interpretation of system (3) put forward in § 3. It is in fact sufficient to find periodic curves K.

For this purpose two methods have been used.

The first is to study, for a given value of γ, all the curves K that meet the level-line

$$Q = 0;$$

that is, the boundary of the region q_γ in the RZ-plane, and by following them by continuity find those that have their other extremity on one of the branches of that same line $Q = 0$. Let K in fact be such a curve, and A and B its two extremities, situated on the line $Q = 0$; further let T be a corresponding trajectory in space, and A' and B' two points on it corresponding to A and B.

In the point A' the angle θ will be $+90°$ or $-90°$, because $Q = 0$, that is to say, the trajectory in that point is normal to the plane through the point and the Z-axis. As the magnetic field is a function of R and z only, that plane will be a plane of symmetry for the trajectory. In the same way the plane through the Z-axis and the point B' will be a plane of symmetry, and then after passing through B' the corpuscle will follow a branch $B'A''$ symmetrical with $B'A'$, then a branch $A''B''$ symmetrical with $A''B'$, and so on; that is to say, we shall have a periodical trajectory.

The corresponding curve K will have stopping points (points d'arrêt) in the points A and B; and when the point P follows the trajectory the point p will go from A to B, then from B to A along the same curve K, then from A to B again, and so on.

The second method consists in studying all the curves K that intersect the R-axis orthogonally. If we then find a curve intersecting the R-axis orthogonally in one point more, this curve will be symmetrical with the R-axis and quite closed, and consequently the trajectory in space will be periodical. This is a point of view used in Darwin's work on the periodical trajectories in the problem of the three bodies.

When a periodical trajectory corresponding to a certain value of γ has been found, other periodical trajectories of the same family can generally be found for γ near this value, and then by variation of γ even closed trajectories. In fact, if Φ is the difference of the values of the angle ϕ for the two points A' and B' (ϕ counted in radian), it is sufficient to vary γ so that the quotient $\Phi : \pi$ becomes a rational number.

With regard to the values of γ giving periodic trajectories, it is easy to see, by looking at the fields of force q_γ, that there are no such trajectories when, for instance,

$$\gamma > -0.5.$$

For $\gamma = -0.8$ there are, and of course there exists a value of γ between -0.8 and -0.5, so that there are periodic trajectories for $\gamma < \gamma'$ but not for $\gamma > \gamma'$. For γ less than -1, the periodic trajectories can only exist in the inner part of the region $Q\gamma$.

The simplest of the periodic curves K are those that connect the two sides of the défilé when γ lies between -1 and γ'. The corresponding trajectories in space have an undulating form, and become the circle with radius 1 lying in the XY-plane with its center in the origin, when γ becomes equal to -1.

In Fig. 15 is seen such a curve K corresponding to $\gamma = -0.8$, and the corresponding trajectory in space is marked with III (in vertical and horizontal projection) in Fig. 16.

The other trajectories in the same figure correspond to $\gamma = -0.97$ and -0.999. These trajectories and the corresponding asymptotic trajectories, etc., are more carefully studied in a paper published in 1911.*

* "Sur une classe de trajectoires remarquables," etc. ("Archiv for Mathematik og Naturvidenskab," Vol. XXXI; Christiania, 1911.)

Another interesting periodic trajectory, corresponding to $\gamma = -0.999$, is seen in Fig. 18 and the corresponding K-curve in Fig. 17.

In Fig. 19 is seen a periodic trajectory lying in the XY-plane and corresponding to $\gamma = -1.2032$. The dotted circles have a radius equal to unity. The corresponding K-curve is the segment of the R-axis contained in the inner part of the region q_γ. The coördinates of this trajectory can be expressed by elliptic integrals, as is the case with every trajectory lying in that plane.*

It would be rather interesting to compute a great number of periodic trajectories of different families. Their shape is sometimes extremely curious, and they can be found rather easily.

Their theory is of much interest in their application to periodic magnetic disturbances.†

* See my Geneva paper of 1907, § 20.

† See "Comptes Rendus," October 1, 1906; Paris.

II

APPLICATION TO AURORA

10. *Explanation of some of Professor Kr. Birkeland's experiments.*

As it has been mentioned in § 1, Professor Kristian Birkeland has made some extremely interesting experiments with a magnetic sphere exposed to cathode rays. A great many new experiments of this kind have been published in the first section of his work, "The Norwegian Aurora Polaris Expedition," 1902–1903,* and more will follow in the next section of that fundamental work.†

As it is well known, the magnetic field due to a uniformly magnetized sphere is identical outside the sphere with the field of an elementary magnet placed in the center of the sphere.

It is of course to be expected that the physical experiments and the mathematical theory will be in accordance, and a close comparison of the results of the two has also hitherto shown the most excellent coincidence.

In another paper the detailed comparison between the theory and the experiment will be fully described, so that I will not here enter into details. Only some of the most striking coincidences ought to be mentioned.

The first thing to be fixed for the application to Professor Birkeland's experiments is our unit of length,

$$\sqrt{\frac{M}{H\rho}}$$

* Obtainable from Longmans, London, and Longmans, Green & Co., New York.

† See "Orages magnétiques et aurores polaires," by Kr. Birkeland. ("Archives des sciences physiques et naturelles," Geneva, 1911.)

centimeters. Here M is the magnetic moment of the sphere, and $H\rho$ the characteristic product of the cathode rays in question; both can be determined by experiments.

It is necessary to observe the relative position of the cathode with regard to the sphere and its magnetic axis, and to take into account the form of the vacuum tube in order to obtain an exact idea of what is to be expected. If, in fact, the interior of the tube is too small, many of the possible trajectories will not reach the sphere, but will strike the interior walls of the tube.

The region of space Q_γ, out of which trajectories could not come, can be seen in Fig. 20, and at the side is seen the corresponding region Q_γ.

The patches where the cathode rays strike the sphere are also in accordance with the calculated trajectories of the simplest shape ($-0.93 < \gamma < 0$), as shown in Fig. 21.

In my paper, "Sur une classe de trajectoires remarquables," * etc., I have shown that the remarkable series of congruent precipitations that are sometimes to be seen on the magnetic sphere is also in full accordance with the theory.

The luminous ring sometimes seen in the magnetic equator of the sphere may correspond to the trajectories in the vicinity of the circular orbit of radius $\sqrt{\frac{M}{H\rho}}$ centimeters (*i.e.* radius equal to our unit) in the XY-plane; but, as I have recently pointed out,† a ring may also be produced by negative corpuscles thrown out from the sphere in the neighborhood of its magnetic equator, as in my models of the solar corona.‡

* See "Orages magnétiques et aurores polaires," by Kr. Birkeland. ("Archives des sciences physiques et naturelles," Geneva, 1911.)

† See "Critique et développements relatifs au mémoire de M. Richard Birkeland," etc. ("Archives des sciences physiques et naturelles.")

‡ See "Sur la structure de la couronne du soleil." ("Comptes Rendus," September, 26 1910; Paris.)

The theory of artificial auroral belts will be more fully developed in the application to aurora. For further application I must refer to my Geneva paper now in preparation.

11. *Application of the region Q_γ to find the auroral regions on the earth.*

In the application to aurora we have considered, as already stated, in § 1, the Gaussian expansion of the magnetic potential of the earth for a point outside it, and have rejected all terms of the series except the first principal term, which then gives us the magnetic moment of the earth and the direction of the magnetic axis. For this axis we take as a definition an earth-diameter parallel with that direction.*

In the applications in the Geneva paper of 1907 I have chosen as the magnetic moment of the earth

$$M = 8.52 \times 10^{25}.$$

The point of intersection of the magnetic axis (the south end) with the surface of the earth is marked on the chart, Fig. 24, for the years 1700 and 1900. The positions are calculated by the formula of Carlheim Gyllensköld.†

After that the unit of length,

$$\sqrt{\frac{M}{H\rho}} \text{ centimeters}$$

is to be found, and it is then necessary to know the product $H\rho$ for the electrified corpuscles supposed to be the cause of aurora.

Regarding this product $H\rho$, we can only make assumptions until further evidence has been obtained. In my Geneva

* It is possible that an axis parallel to this one and suitably chosen might be still better.

† For the details, see my Geneva paper, 1911–12, Part I.

paper of 1907 I have calculated the unit of length corresponding to cathode rays, β-rays and α-rays of radium. The result was as follows:

	$H\rho$	Unit of Length in Kilometers
Cathode rays	108	8.9×10^6
	543	4×10^6
β-rays	1,801	2.2×10^6
	4,524	1.4×10^6
α-rays	291,000	1.7×10^5
	398,000	1.46×10^5

The dimensions of the regions Q_γ are therefore immense as compared with those of the earth, as will be seen from Fig. 22.

This figure represents the shapes of a series of regions q_γ in the vicinity of the origin. The scale, which is much larger than on Plate I, is marked on the R-axis, $\sqrt{\frac{M}{H\rho}}$ being taken as the unit of length. The parts q_γ, which are white, are not continued up to the origin, in order to avoid indistinctness. The five dotted circles indicate the relative size of the earth as compared with the spaces Q_γ for various kinds of corpuscles; the innermost circle corresponds to cathodic rays where $H\rho = 315$, the two next to β-rays where $H\rho = 2891$ and 4524, and the two outer circles correspond to α-rays where $H\rho = 2.91 \times 10^5$ and 3.98×10^5.*

Now the first necessary condition fulfilled by corpuscles sent out from a point at a distance from the earth greater than $\sqrt{\frac{M}{H\rho}}$ and reaching the earth, is that the corresponding space $Q\gamma$ extends without interruption from the point of emanation to the earth. The constant γ corresponding to the trajectory cannot therefore be less than -1. On the

* See my Geneva paper of 1907, § 17.

other hand, a detailed study of the shapes of the regions q_γ for $\gamma > 0$ shows that γ cannot be greater than

$$\left(\frac{\Delta}{2}\right)^3$$

where Δ is the distance from the center of the earth to the aurora measured with our unit of length $\sqrt{\frac{M}{H}}$.

The regions of the earth in which the corpuscles can strike the atmosphere will thus be confined to two zones round the magnetic axis and limited by circles whose radius in degrees is easily found.* If we call that radius Ω, we have, with sufficient exactness,

$$\sin \Omega = \sqrt{2\,\Delta}$$

Δ being the above-mentioned distance. If we measure Δ in centimeters, and if it is equal to D centimeters, we obtain

$$\sin \Omega = \sqrt{2\,D\sqrt{\frac{H\rho}{M}}}.$$

Thus for cathode rays I found Ω to be between 2° and 4°, for β-rays between 4° and 6°, and for α-rays between 16° and 19°.

The corresponding regions on the earth are seen in Fig. 23, *a* and *b*.

Here we see the first objection to the theory: the radius of the auroral zone is too small for cathode-rays and β-rays of the known kinds. In fact the real auroral zone is generally limited by a circle of radius about 23°, and sometimes goes much farther from the magnetic axis.

We will return to this important question later on.

* See my Geneva paper of 1907, §§ 6 and 17.

12. *Application of the trajectories through the origin to find the auroral belts.*

We will now make the further hypothesis that the corpuscles come from the sun, and see if that hypothesis will reduce the theoretical auroral regions still more.

If we suppose, as in my Geneva paper, that

$$100 \leqq H\rho \leqq 400{,}000,$$

it will be seen that only the corpuscles whose trajectories lie in the vicinity of those through the origin can reach the earth; the others return into space.

The study of the trajectories through the origin is therefore most important for the application to the aurora.

Now the angle between the plane normal to the earth's magnetic axis and the line from the earth to the sun varies between $-35°$ and $+35°$, and therefore trajectories whose infinite branches come from directions outside this interval have to be excluded.

As the computations show,* these excluded trajectories correspond to γ between 0 and -0.2, and their point of intersection with the earth will lie in regions round the magnetic axis limited by circles.

Only two belts of the theoretical aurora region found in the foregoing paragraph will thus be left. The breadth of these belts will be still more reduced when we only take into account auroras which are visible when the sun is below the horizon; for γ will then be confined to the interval from about -0.5 to -1.

The auroral belts are thus explained, but, as has already been pointed out, we do not get the real situation of these belts; they are too near the magnetic axis for hitherto known corpuscles. But the fact that the magnetic axis is

* See my Geneva paper of 1907.

in the middle of the belts is in accordance with reality, as may be seen on the chart of the frequency of aurora borealis (Fig. 24).

13. *Explanation of a series of peculiarities regarding aurora, as an application of the theory of the trajectories through the origin. Formation of auroral curtains.*

Let us now suppose that corpuscles are sent out from a point of the sun's surface in all directions into space. Let us further assume that the constant $H\rho$ is the *same* for all these corpuscles and that it has a value between 100 and 400,000.

As I have pointed out in the foregoing paragraph, the corpuscles whose directions of emanation are very nearly tangent to a trajectory through the origin will reach the earth, the others will pass by. Let us call the directions tangent to trajectories through the origin, *distinguished directions*.

Their configuration and their number vary enormously with the relative position of the point of emanation with regard to the magnetic axis of the earth. Now this position is continually changing because the magnetic axis follows the movements of the earth, and consequently the conditions for the occurrence of aurora must vary considerably with time, a circumstance which accords well with the sudden variable character of the auroral phenomena.

This may also explain the fact frequently observed, that aurora occurs on two consecutive days almost at the same hour; in fact, the relative position between the point of emanation and the earth's magnetic axis is repeated after twenty-four hours.

But as the point of emanation follows the sun's rotation, we have another well-known period of twenty-seven days

between two consecutive passages of a sun spot through the solar meridian whose plane passes through the earth. The cases in which the sun spot does not give rise to a new aurora the next time it passes may be explained by the non-coincidence of the directions of emanation with the distinguished directions; in fact a slight difference in the relative positions can cause the distinguished directions that existed at the first passage to disappear.

Let us now assume that a beam of corpuscular rays is sent out with the same velocity from a surface of emanation on the sun, and that it reaches the earth's atmosphere and produces aurora. The constant γ for the different trajectories in the beam is given by the formula

$$2\gamma = R \sin\theta - \frac{R^2}{r^3},$$

where R and r are coördinates of the point from which the trajectory starts, and θ is the angle between its tangent at that point and the plane through the magnetic axis of the earth. It is clear that if we choose the same γ for all the trajectories, the beam will consist of almost parallel rays.

All the trajectories will then be in the interior of the region Q_γ, and consequently the aurora also. Now near the earth, this region Q_γ is the very narrow space inclosed between the two surfaces of revolution corresponding to $k = +1$ and $k = -1$, and the aurora will therefore appear in the region of the atmosphere between these two surfaces. That region extends all round the earth with the magnetic axis in the middle, and is very narrow. For instance, the thickness (see Plate XX and Figs. 22 and 25) is

For cathode rays, between	3	and	20 meters;
For β-rays, between	50	and	150 meters;
For α-rays, between	9,000	and	13,000 meters.

Therefore, as already pointed out in my paper, "Sur le mouvement d'un point matériel portant une charge d'électricité sous l'action d'un aimant élémentaire" (Christiania, 1904), the rays of the beam distributed in this region may give rise to the light phenomenon which we call, according to circumstances, an arc or an auroral curtain.

We will study more closely a case in which the spreading out of a cylindrical beam into a curtain can be explained mathematically.*

In Fig. 26 the earth is situated in the origin, with its magnetic axis coinciding with the Z-axis and the north polar region turned upward. Let M_γ be the point of emanation of a corpuscle following the trajectory through the origin corresponding to the constant γ. Let D_γ be the tangent to that trajectory at the point M_γ, ψ_γ the angle between the radius vector to M_γ and the XY-plane, and Φ_γ the variation of the angle ϕ when the corpuscle moves from M_γ to the origin. Φ_γ is positive or negative according to increasing or decreasing angle ϕ.

Let us give the point M_γ a little displacement without changing the distance from the origin, and at the same time vary γ continuously so that we get the corresponding trajectories through the origin in the new position.

Let us call $\Delta\psi$ and $\Delta\phi$ the augmentations of ψ_γ and ϕ corresponding to the displacement of M_γ, and $\Delta\gamma$ and $\Delta\Phi$ the corresponding augmentations of γ and Φ_γ.

Here $\Delta\gamma$ is independent of $\Delta\phi$ and

$$\Delta\Phi = \Delta\phi + \Delta_1\Phi,$$

where $\Delta_1\Phi$ is the augmentation of Φ_γ when ϕ is constant and ψ_γ varies from ψ_γ to $\psi_\gamma + \Delta\psi$.

To find $\Delta\gamma$ and $\Delta_1\Phi$ it is sufficient to know how the angles

* See my Geneva paper of 1907, § 19.

ψ_γ and ϕ_γ vary with γ, for the trajectories through the origin.

Let us now study the variation of the point of precipitation of the corpuscle upon the earth and its displacement corresponding to the displacement of the point M_γ.

Let us consider the point of intersection A_0 of the trajectory from M_γ with a sphere Σ concentric with the earth, and whose radius D is equal to the distance D from the center of the earth to the aurora. Let A be the displaced point corresponding to the new position of M_γ. Further, let C_0 and C be two smaller circles on Σ with their centers in the magnetic axis, and passing through A_0 and A respectively. Let MA_0 and MA be two great circles through A_0 and A_1 and M, and let A' be the point of intersections of MA with the circle C. The position of A is determined by A_0A' and $A'A$, and we will find these displacements as functions of $\Delta\gamma$ and $\Delta\Phi$.

As γ is negative, let us put

$$\gamma = -\gamma_1$$

and let

$$\sqrt{\frac{M}{H\rho}} = c.$$

We then have

$$\sin\alpha = \sqrt{\frac{2\gamma_1 D}{c}}$$

and

$$\Delta\alpha = -\frac{1}{\cos\alpha}\sqrt{\frac{D}{2c\gamma_1}}\cdot\Delta\gamma,$$

whence

$$A'A = -\frac{D}{\cos\alpha}\sqrt{\frac{D}{2c\gamma_1}}\cdot\Delta\gamma.$$

Here A will be nearer or farther from the magnetic axis than A_0 corresponding respectively to positive or negative $\Delta\gamma$. To find A_0A' we may remark that the angle A_0MA'

is equal to $\Delta\Phi$; if $\Delta\Phi$ is then measured in degrees, we have

$$A_0A' \frac{\pi}{180}[\Delta\phi + \Delta_1\Phi] D \sin\alpha.$$

Here A' will be to the west of the point A_0, if A_0A' is positive, and to the east if it is negative.

By these formulæ the situation of the point A relative to A_0 can be calculated; here A_0 is the point of precipitation of the corpuscle coming from M_γ, and A the point of precipitation of the corpuscles coming from the displaced point M_γ.

It is clear that we can now find the precipitations of all the corpuscles coming from a whole surface of emanation and corresponding to continuous variations of γ, and this, as we shall see, will give a very natural explanation of an auroral curtain.

If the situation of the point of emanation M_γ is chosen in such a manner that $\Delta\Phi$ is very great and $\Delta\gamma$ very small compared with $\Delta\psi$, then A_0A' will be very long compared with $A'A$. The corpuscular rays sent out from the surface of emanation will therefore be spread out like a fan in approaching the earth, and will strike the atmosphere as an auroral curtain consisting of beams * along the lines of force, as on the photographs on Plates XXIII, XXIV which I took at Bossekop in March, 1910.

Situations like this occur for the values of ψ_γ corresponding to a maximum or a minimum of the function

$$\psi_\gamma = f(\Phi_\gamma).$$

In Fig. 28 we see a curve like this constructed by means of computed trajectories through the origin.

The curve is to be continued to the right, and *will tend more and more to consist of an infinite series of congruent arcs*, like

* For details concerning the trajectories in such a beam, see my Geneva paper of 1911–12, § 25.

a sinusoid, giving an infinite number of trajectories corresponding to values of γ tending towards the limit γ^* mentioned in § 8.

A detailed computation of an auroral curtain corresponding to the minimum for

$$\gamma = -0.928934$$

will be found in my Geneva paper of 1907. The result was that the length of the curtain was more than a thousand times as great as its thickness, which is in good accordance with the reality.

Even the remarkable fact that we can have *several auroral curtains one behind another* † can very easily be explained. There may be two reasons.‡ If the surface of emanation sends out corpuscles whose velocity has a finite number of distinct values differing very little from one another, our unit of length will differ for the different kinds of corpuscles, and therefore also the distance of the curtain from the magnetic axis; we shall get curtains one behind another.

But even with the same kind of corpuscles we may get series of curtains. Let us, for instance, assume that the direction of emanation corresponds to a value like γ^*, which is the limit for a series of values of γ corresponding to trajectories going round the earth an increasing number of times (see § 8). Then the beam sent out from the surface will give trajectories corresponding to several cases of fan-spreading and to corresponding values of γ differing very little. Consequently a series of auroral curtains will be spread out all round the magnetic axis in the north and south polar regions *at one time*, and if they all occur in the same meridian, they

* For details concerning the trajectories in such a beam, see my Geneva paper of 1911–12, § 25.

† See Plate XXV, which represents a photograph taken by me on February 28, 1910, in Bossekop.

‡ See my Geneva paper of 1907, § 19.

will appear one behind another. A value of ψ_γ like this is indicated in Fig. 29, which is the continuation of the curve

$$\psi_\gamma = f(\Phi_\gamma),$$

where the line corresponding to the value ψ_γ in question intersects the curve in an infinite number of points near the minima, giving curtains whose corresponding angles are found on the axis of abscissæ. The situations giving rise to curtains and series of curtains will be rapidly passed over because the magnetic axis of the earth is rotating round the axis of rotation by the diurnal motion, which changes the relative position of the surface of emanation assumed to be on the sun. This is in accordance with the observed fact, namely, that these beautiful phenomena come suddenly and last only a very short time.

III

OBJECTIONS TO THE PRECEDING THEORY. INVESTIGATION UNDER MORE GENERAL ASSUMPTIONS

14. *The position of the auroral zone.*

As we pointed out in § 2, the above theory was only the first approximation to reality corresponding to the simplifying hypothesis there set forth. It is therefore rather remarkable to see how many peculiarities of the aurora can already be explained. But there are also facts that do not agree with the developed theory.

The chief of these, already pointed out by Villard, is the real situation of the zone of maximum frequency of aurora. We have seen that at its outer border the auroral zone was limited by a circle, whose angular radius Ω was given by the formula

$$\sin \Omega = \sqrt{2\, D \sqrt{\frac{H\rho}{M}}},$$

which gave only about 6 degrees for β-rays corresponding to $H\rho = 5000$, and about 18 degrees for α-rays of radium. The real value corresponding to the general situation of the maximal zone of aurora borealis is, on the contrary, 23°, corresponding to $H\rho$ about a million.

In my Geneva paper of 1907 I already admitted the possibility that the corpuscular rays corresponding to large sun spots and brilliant corresponding aurora may have a much larger value of $H\rho$ than those corresponding to aurora in the maximal zone; but I did not pay much attention to that side of the question. I thought it possible that if

we changed hypothesis V, and took into account the real Gaussian expression for the magnetic field with all the terms hitherto considered, we might also find the real situation of the auroral zone.

To test this was an extremely laborious and tedious task. I had first to compute tables and draw corresponding plates for taking out graphically the components of the earth's magnetism in each point of space required. To calculate these components would have been impracticable, because each contains about fifty terms in its expression. I had then to calculate the lines of force and then the trajectories.

These computations have been given in detail in my recent Geneva paper of 1911–12. The result, however, was negative, being as follows :

It seemed probable, but could not be decided with certainty, that the consideration of the Gaussian expression with all terms known could not explain the real situation of the auroral zone.

In the meantime Professor Kr. Birkeland had published a note in the " Comptes Rendus," Paris,* where he explained the situation of the auroral zone by assuming for the corpuscles in question hitherto unobserved properties — first, that the velocity is very nearly the velocity of light (an assumption that gives a sufficiently high value of $H\rho$), and next, that the charge is negative and that the corpuscular rays have such great power of penetration that they can reach almost down to sea level, a penetration equivalent to that required to pass through 760 millimeters of mercury. It is clear that if this hypothesis could be verified, we should have in the auroral rays corpuscular rays of extreme interest.

Now the first exact measures of the altitude of aurora, by a photographic method, obtained on my auroral expedi-

* January 24, 1912.

tion * to Bossekop in February and March, 1910, gave a series of reliable determinations of the altitude of the lower edge of the auroral curtains; and it was shown that these auroræ did not possess anything like the extreme power of penetration supposed by Birkeland. Some of them, however, which reached the lower limit of about 40 kilometers, gave a penetration greater † than the β-rays of radium, for which $H\rho = 5000$.

The 120-and-odd exact measurements of aurora photographed simultaneously in Christiania and at Aas on April 8, 1911, also proved ‡ that the beams and draperies stopped at an altitude of about 60 kilometers, which corresponds to a penetration equal to that of β-rays; and this too is not in accordance with Birkeland's views. It is possible that the great auroræ sometimes seen during the maximal period of solar activity may reach as far down as Birkeland assumes, but this has yet to be proved.

In order to explain the situation of the auroral zone when $H\rho$ is only of the same order as for β-rays, I endeavored to find out whether the outer magnetic field caused by corpuscular rays round the earth could have an influence on the situation of the auroral zone and, for instance, draw it away from the magnetic axis when the quantity of corpuscles was large enough, as during magnetic storms.

The results of the computations were favorable to this hypothesis, as we shall see in the next paragraph.§

* See "Bericht über eine Expedition nach Bossekop," etc. ("Videnskabsselsk. Skr.," 1911; Christiania.

† See P. Lenard: "Ueber die Strahlen der Nordlichter Sitzungsberichte der Heidelberger Akademie der Wissenshaften," 2 Juli, 1910; and "Ueber die Absorption der Nordlichtstrahlen en der Erdatmosphäre," *ibid.*, 13 Mai, 1911.

‡ See my Geneva paper of 1911–12, § 27.

§ For the first results see, "Sur la situation de la zone de fréquence maximum des aurores boréales d'après la théorie corpusculaire." ("Comptes Rendus," Paris, October 24, 1910.)

15. *The action of the outer corpuscular magnetic field on the situation of the auroral zone. General problem.*

As shown by Professor Kr. Birkeland's researches on magnetic storms, it seems extremely probable that magnetic storms are due to the action of large corpuscular currents in space outside the earth. If we admit that such currents have electromagnetic action, it is clear that the normal terrestrial field in space can be completely altered by these corpuscular currents, especially at a great distance from the earth, where the terrestrial field is very weak.

The perturbation of the terrestrial field in space will vary with the amount of corpuscles and with their trajectories. On the other hand, each corpuscle will have an electromagnetic action on every other corpuscle and on the electromagnetic field of the earth, and vice versa. This brings us to the following difficult problem.

A number of celestial bodies — *e.g.* the planets and the sun — are moving in a given manner in space. These bodies may be magnetizable and surrounded by electromagnetic fields. On the other hand, we assume a number of corpuscles moving in space, and at a fixed moment we suppose the motion of all these corpuscles to be known. The motion and the electromagnetic field at any given future moment have then to be found.

It is clear that this problem must be extremely difficult. Each corpuscle exerts an electromagnetic action on every other corpuscle, and we have reciprocal electromagnetic action between the corpuscles and the celestial bodies. Further, the electromagnetic action is propagated in space with the velocity of light, so that the field in any point is a function of the conditions in space, not only at the moment immediately preceding, but for a definite period before. It

seems probable, therefore, that the equations defining the development of the phenomena will not be differential equations, but rather integral equations, where the unknown quantities are put under signs of integration, as in the case of the Fredholm integral-equations.

We cannot now treat this general problem. We will only mention a simple case. In the discussion of the trajectories shown in the wire model (Fig. 6), we saw that there was a large stream of corpuscles going round the magnetic sphere on the evening and night side, if we consider the sphere as the earth and the surface of emanation as the sun. This stream, as we saw, can go quite round the earth and form a corpuscular ring in the magnetic equatorial plane, with radius equal to our unit of length $M^{\frac{1}{2}}(H\rho)^{-\frac{1}{2}}$ centimeters.

If we suppose the outer magnetic field to be due only to a corpuscular ring such as this, it is possible to find the regions of space out of which trajectories cannot come. In my Geneva paper I have discussed in detail an ideal case such as this.

I will here only give the principal results.

Let us assume a corpuscular current to be lying in the magnetic equatorial plane, having the form of a circle with radius

$$\sqrt{\frac{M}{H_1\rho_1}} \text{ centimeters.}$$

Here M is the magnetic moment of the earth, and $H_1\rho_1$ the characteristic product of the corpuscles in the ring. If the charge of these corpuscles is negative, the corpuscles will be supposed to be moving in a direction from west by south to east; if the charge is positive, in the opposite direction, all in accordance with the trajectories in the earth's normal field.

Let us further denote with H the direct magnetic action of this ring observed on the surface of the earth. If this action

is only about 30 units γ (one γ equal to 10^{-5} C.G.S.), it will be sufficient to draw the auroral zone corresponding to other corpuscles so far away from the magnetic axis that we shall get the real position of the observed zone of maximal frequency, as will be seen from the following table,* where $H_0\rho_0$ is the characteristic product for the aurora corpuscles, $H_1\rho_1$ the same for the corpuscles in the ring.

Table of the Values of H if the Auroral Zone is drawn down to the Observed Zone corresponding to $\Omega = 23°$

$H_1\rho_1 =$	$H_0\rho_0 = 10^2$	$10^{2.5}$	10^3	$10^{3.5}$	10^4	$10^{4.5}$	10^5	$10^{5.5}$	10^6
10^2	33	33	33	33	33	32	30	23	4
$10^{2.5}$	32	32	32	32	32	31	29	23	4
10^3	32	32	32	32	32	32	30	23	4
$10^{3.5}$		32	32	32	32	32	30	23	4
10^4		79	32	32	32		30	23	4
$10^{4.5}$		460	140	46		32		23	4
10^5				250	79	32	30	22	4
$10^{5.5}$				1400	450	140	43	22	4
10^6					2600	790	230	58	3
$10^{6.5}$						4400	1300	340	21

Thus a ring consisting of β-rays can bring the theoretical auroral zone for cathode rays down from $\Omega = 3°$ to $\Omega = 23°$, when the current in the ring becomes large enough. As I have shown in my Geneva paper, the current will then be of the order of a hundred million amperes, and its distance from the earth about one million kilometers.

I think it probable that there is such a ring or at least a large stream of corpuscular rays bending round the afternoon and night side of the earth, with an action similar to that of a ring.

Another effect of an exterior magnetic field such as this is to concentrate the auroral zones corresponding to different

* See my Geneva paper of 1911–12, § 19.

kinds of corpuscles in one belt corresponding to $\Omega = 23°$ in the above example, which may give an explanation of the observed fact that in the maximal zone we have auroræ of very different degrees of penetration. During my expedition, I measured auroral curtains whose lower edges were about 40, 50, 60, 70, 80, and 125 kilometers above the ground.

The hypothesis of the outer disturbing field that draws the auroral zone away from the magnetic axis has a most interesting application to *aurora observed during magnetic storms*.

It is a well-known fact that auroræ seen in lower latitudes *e.g.* in the middle of Europe — are *always* accompanied by magnetic storms, which is not generally the case with auroræ in the zone of maximal frequency.

On the other hand, auroras occurring far away from the magnetic axis do not penetrate so far into the atmosphere as the auroræ in the maximal zone. An example of this is afforded by my photographic parallax measurements of the auroral curtains at Bossekop and in Christiania, the lower limit of the altitude found at Bossekop being 40 kilometers, and in Christiania, on April 8, 1911, about 60 kilometers. It seems probable that the most penetrating auroræ correspond to the greatest values of the constant $H_0\rho_0$, and we then have a contradiction when we calculate the outer border of the auroral zone by the formula

$$\sin \Omega = \sqrt{2\, D \sqrt{\frac{H_0\rho_0}{M}}}$$

which corresponds to no action of an exterior field.

But if we take such a field into account, we can have the aurora even when caused by cathode corpuscles drawn away from the magnetic axis into positions corresponding to the observed facts; but the action H, which was negligible in

the case of auroræ in the maximal zone, now grows very fast, as may be seen from the following Table.

$\Omega = 30°$	$35°$	$40°$	$45°$	$50°$
$H = 140\gamma$	320γ	660γ	1200γ	1900γ

where the corpuscles in the ring have a product $H_1\rho_1$, equal to or greater than the product $H_0\rho_0$ of the aurora corpuscles.

We have here an explanation of the fact that such auroræ must be accompanied by magnetic storms. The action H will probably be concealed behind the greater disturbances caused by portions of the corpuscular system, of which the ring is only a part.

We have here a wide field for investigation.

16. *Research based upon the hypothesis that the sun is surrounded by a magnetic field, and that the corpuscles are influenced by gravitation, pressure of light, and electrical attraction to, or repulsion from, the sun.*

As already pointed out in the introduction, the next step in auroral research would be to substitute for hypothesis II the assumption that the sun is surrounded by a magnetic field.

When I constructed the first wire models of the trajectories through the origin, in 1906, I was even then struck by the resemblance between these trajectories and the swallow-tailed streamers of the solar corona during years of minimum frequency of sun spots. I did not, however, follow up the suggestion until 1911, when I published a note on the solar corona, accompanied by figures of wire models representing trajectories of electric corpuscles sent out normally from the surface of a magnetic sun.*

* See "Sur la structure de la couronne du soleil." ("Comptes Rendus," Paris, February 20, 1911.)

We reproduce here two of these figures (Figs. 30 and 31) to show the resemblance to the solar corona observed on the American expeditions * (Figs. 32 and 33).

For details the reader is referred to the note in question. One point, however, may be mentioned. If we succeed in identifying the trajectories with corona streamers, this would supply a method of finding *the magnetic moment of the sun, if we know the product $H\rho$ of the corpuscles*, and *vice versa.*

On the other hand, a detailed study of the trajectories in the sun's equatorial plane, such as I gave in § 20 of my Geneva paper of 1907, gives, as Professor Birkeland † has pointed out, an explanation of the interval of about forty hours separating the passage of a sun spot over the central meridian of the sun's disk, and a subsequent magnetic storm. It also gives a relation between $H\rho$ and the sun's magnetic moment.

Besides the hypothesis of a magnetic sun, there are other assumptions that can be made. According to the auroral theory of Arrhenius, for instance, the sun sends out small electrified material particles of about one ten-thousandth part of a millimeter pushed away by the pressure of light. As Arrhenius further supposes the sun to be magnetic and to have an electric charge, one is led to study the motion of an electrified corpuscle that is influenced by the following forces:

(1) The field of an elementary magnet,
(2) Gravitation,
(3) The pressure of light, and
(4) Electric attraction or repulsion.

* Publications of the United States Naval Observatory, Vol. IV, Appendix I.

† See "Comptes Rendus," January 10, 1910 (Deslandres) and January 24, 1910 (Birkeland).

The last three forces are supposed to emanate from the elementary magnet, and to be inversely proportional to the square of the distance.

I have already treated this problem in a paper * published in 1907, and have given further results in a note in " Comptes Rendus " † in 1911. The chief results are here given.

If we put the origin of a Cartesian system of coördinates in the elementary magnet and the z-axis along its axis, the equations of motion will be

$$\frac{d^2x}{dt^2} = \lambda M\left(\frac{3\,yz}{r^5}\frac{dz}{dt} - \frac{3\,z^2 - r^2}{r^5}\frac{dy}{dt}\right) + \mu\frac{x}{r^3},$$

$$\frac{d^2y}{dt^2} = \lambda M\left(\frac{3\,z^2 - r^2}{r^5}\frac{dx}{dt} - \frac{3\,xz}{r^5}\frac{dz}{dt}\right) + \mu\frac{y}{r^3},$$

$$\frac{d^2z}{dt^2} = \lambda M\left(\frac{3\,xz}{r^5}\frac{dy}{dt} - \frac{3\,yz}{r^5}\frac{dx}{dt}\right) + \mu\frac{z}{r^3},$$

where M is the moment of the elementary magnet, and λ and μ are constants depending on the intensity of the acting forces and of the nature of the corpuscle. From this we get the two first integrals,

$$v^2 = -\frac{2\,\mu}{r} + C$$

and

$$x\frac{dy}{dt} - y\frac{dx}{dt} = a + \lambda M\frac{R^2}{r^3},$$

where v is the velocity, and a and C constants of integration.

Let θ denote the angle between the tangent at a point on the trajectory, and the plane passing through that point and the Z-axis; then we find from the two integrals above, that

$$\sin\theta = \frac{ar^3 + \lambda MR^2}{Rr^2\sqrt{Cr^2 - 2\,\mu r}}$$

* " Sur un probleme relatif au mouvement des corpuscules électriques dans l'espace cosmique." (" Videnskabsselskabets Skrifter," I, 1907; Christiania.)
† *l.c.*, March 6, 1911.

and, as we have seen in the first part of this memoir, the fact that $\sin\theta$ cannot pass beyond the interval -1 to $+1$ defines a region in space

$$-1 \leqq \frac{ar^3 + \lambda MR^2}{Rr^2\sqrt{Cr^2 - 2\mu r}} \leqq 1$$

out of which the trajectory cannot come, a circumstance which is extremely useful in the discussion of the trajectories.

I treated fully, several years ago, the different forms of these regions, but the paper has not yet been published. We can have regions formed like hollow rings * corresponding to stable trajectories in their interior. At the limit the rings contract into circles with their centers on the Z-axis and situated in, or parallel with, the XY-plane.

These circular trajectories were found directly in my paper of 1907. If we put

$$x = r_0 \cos\psi_0 \cos\left(\frac{vt}{r_0 \cos\psi_0}\right),$$

$$y = r_0 \cos\psi_0 \sin\left(\frac{vt}{r_0 \cos\psi_0}\right),$$

$$z = r_0 \sin\psi_0$$

in the equations of motion, and determine the constants r_0, ψ_0 and v so that the equations are satisfied, we find,

(1) Circles in the XY-plane corresponding to the condition

$$v^2 r_0^2 + \mu r_0 + \lambda M v = 0, \qquad \text{and}$$

(2) Circles parallel to that plane corresponding to the conditions

$$\cos\psi_0 = \frac{2\mu^2}{9\lambda M v^3},$$

$$r_0 = -\frac{2\mu}{3v^2}.$$

* See also my two notes in the "Comptes Rendus," Paris, February 10 and 17, 1913.

Here μ must be negative and

$$2\mu^2 < 9\lambda M v_3.$$

In a note of March 6, 1911, I have further shown that the differential equations can be reduced to the following system, in analogy to what is found in the case of motion in the field of an elementary magnet alone.

$$\frac{d^2R}{dt^2} = \frac{1}{2}\frac{\partial Q}{\partial R}, \qquad \frac{d^2z}{dt^2} = \frac{1}{2}\frac{\partial Q}{\partial z},$$

$$\left(\frac{dR}{dt_0}\right)^2 + \left(\frac{dz}{dt}\right)^2 = Q,$$

where

$$Q = C - \frac{2\mu}{r} - \left(\frac{ar^3 + \lambda M R^2}{Rr^3}\right)^2.$$

This system can be interpreted in the same manner as that described in § 5, which gives very suggestive methods for discussing the trajectories. When R and z are found from this system, then the rest of the integration can be performed by quadrature.

In the XY-plane, the equations of the trajectories can be found by elliptic integrals; for when R and ϕ are polar coördinates in this plane, we have

$$d\phi = \frac{aR + \lambda M}{R} \frac{dR}{\sqrt{CR^4 - 2\mu R^3 - (aR + \lambda M)^2}}.$$

The detailed study of the trajectories seems also to be of importance for the cosmogony, judging from a recent note by Professor Kr. Birkeland.*

We have here a field which well repays investigation.

*"Sur l'origine des planètes et des satellites." ("Comptes Rendus," November 4, 1912. And my note, "Remarque sur la Note de M. Kr. Birkeland," etc. (*Ibid.*, November 25, 1912.)

17. *The new photographic method for measuring the position and altitude of aurora. Some of the results obtained, and the conclusions relative to the nature of the corpuscles and to the composition and temperature of the upper air.*

In order to obtain an objective impression of the auroral phenomena, I made,* as already mentioned, an expedition to the well-known place Bossekop, in the north of Norway, during February and March, 1910. My purpose was to photograph auroras and to measure their altitude by obtaining parallactic photographs simultaneously from two stations connected by telephone. Both purposes were accomplished with entire success.

Previous to this, the only photograph of aurora with an exposure of less than one minute was one taken by Brendel at Bossekop on February 1, 1892. It represents part of an auroral curtain, and the time of exposure was seven seconds.

By means of a cinematographic lens (aperture, 25 millimeters; focal distance, 50 millimeters) and lumière etiquette violette plates, I succeeded in reducing the time of exposure to an average of two seconds, sometimes to a fraction of a second only.

The complete report of the expedition has been published in " Videnskabsselskabets Skrifter," Christiania,† so that it is not necessary here to enter into details. In this report 342 photographs of aurora are reproduced in their original size of 4×5 centimeters, 24 are enlarged to 11×15 centimeters, and there are 44 pairs of parallactic photographs taken simultaneously from the two stations, Alten church and Upper Alten school, 4300 meters apart. (See Plate XXVI. In the

* As assistant I had the Norwegian meteorologist *Bernt Johannes Birkeland*, member of Roald Amundsen's future north polar expedition.

† "Bericht über eine Expedition nach Bossekop," etc., *l.c.*, 1911.

background of the pictures here reproduced we have the well-known constellation of the Great Dipper, and the altitude calculated from the photographs was from 100 to 120 kilometers. For the details we refer to the full report.)

The measurements of the altitude of aurora are seen graphically in Fig. 34.

We see that most of the altitudes are from about 100 to 120 kilometers, and that the inferior limit is about 40 kilometers, the superior about 350.

Parallactic photographs have since been taken simultaneously in Christiania and in Aas on February 22 and April 8, 1911,* and the results have been carefully discussed in my Geneva paper of 1911–12. We give here the two parallactic photographs reproduced in that paper from the 8th of April at 11 h. 35 m. 30 s. (Central European time). (See Plate XXVII.)

The aurora appeared as beams and curtains, and in the background is the well-known constellation of Perseus.

The great beam in the middle reached from about 370 kilometers down to 76 kilometers, and was at a distance of about 500 kilometers.

The measurements of the altitude of the aurora on April 8 are shown graphically in Fig. 35.

The long beams are here represented by vertical lines, the lowest limits by horizontal short lines, and other calculated points of the aurora by dots.

In the photographic method of measuring the altitude and the situation of the aurora we have for the first time employed an altogether objective and exact method of observing this phenomenon; and it is to be hoped that its

* On this occasion by my assistant on the Bossekop expedition, the meteorologist Bernt Johannes Birkeland, and an assistant at Aas.

systematic use will soon give very important results.* It is especially important to have simultaneous records from series of stations all round the auroral belts in the arctic and antarctic regions; for many of the auroral phenomena, such as curtains and coruscations, need to be studied at several stations simultaneously along the auroral belts.

Parallactic photographs will probably also yield decisive data relating to the nature of the corpuscles and to that of the upper air.

In some previously mentioned papers (see § 14), Professor Lenard has published a very interesting formula for the absorption of cathode rays and β-rays when they come from space down into the atmosphere. He supposes, like Birkeland, that the auroral beams in the curtains are formed by negative corpuscles following magnetic lines of force, and he then finds the following law of absorption:

Let I_0 be the initial intensity of the beam in space outside the atmosphere, and let I be the intensity at a height of h centimeters above the earth's surface, then

$$\log \text{nat} \frac{I_0}{I} = \frac{a}{b \cos \theta} e^{-bh}$$

where

$$b = 0.1238 \times 10^{-5}$$

and where θ is the angle between the beam and the vertical; a is a constant depending on the penetrative power of the corpuscles. In establishing this formula, Lenard supposes the composition of the atmosphere to be the same at all altitudes, as also the mean temperature. He then finds the following diagram for the absorption.

* After having finished this report I made a new expedition to Bossekop in March, 1913, accompanied by the meteorologist B. T. Birkeland. We secured 450 pairs of parallactic photographs with a base of 27 kilometers, which will give excellent results.

From this he draws the conclusions that the laws of absorption of cathode rays and β-rays — that is to say, of negatively charged corpuscles — agree well with the auroral phenomena, and that the assumed β-rays corresponding to the lower limit of altitude, 40 kilometers, must be a more penetrating kind than those observed up to that moment.* On the other hand, the fact that aurora have been measured and found to have an altitude of more than 350 kilometers gives him a proof that the air in this region must consist of very light gases, especially hydrogen, a conclusion which is in accordance with the calculations of the composition of the air at different altitudes by Hann, Humphreys, Jeans, and Wegener.† He indicates finally the interest in calculating the laws of absorption (especially for altitudes of over 100 kilometers) corresponding to the occurrence of these light gases.

For finding these new formulæ of absorption it is also necessary to know the variation of temperature upward. Now, as I have explained in my Geneva paper of 1911–12, the auroral beams can give information regarding the temperature; we may make the following hypotheses:

(1) The atmosphere above 100 kilometers consists of pure hydrogen.

(2) Its temperature above 100 kilometers is constant, equal to $t°$ Centigrade.

(3) An auroral beam, situated entirely in the region above

* M. Danysz has now found β-rays where $H\rho$ is up to 26,000. See his paper, "Sur les rayons β de la famille de radium." ("Le Radium," January, 1912.)

† Hann, "Das Daltonsche Gesetz und die Zusammensetzung der Luft in grossen Höhen." ("Zeitschrift der Oesterr. Gesell. f. Meteorologie," 1875.)

Humphreys, "Distribution of Gases in the Atmosphere." ("Bull. of the Mt. Weather Observ.," 1910.)

Jeans, "Thermodynamics."

Wegener, "Thermodynamik der Atmosphäre," 6 Kapitel.

100 kilometers, consists, as Lenard supposes, of cathode rays or β-rays coming down in straight lines along lines of magnetic force.

(4) The luminous part of the beam corresponds to the part where the intensity diminishes from AJ_0 to BJ_0, A being near unity, B near zero, and J_0 being the initial intensity. I then found that

$$273+t=\frac{1.03\ L}{\log\left(\frac{\log B}{\log A}\right)},$$

where L is the difference of altitude of the upper and lower end of the beam, and the logarithm is the Briggsian with base 10.

In other words, *the auroral beams may serve directly as a thermometer for the upper air.*

In applying this formula for the measured auroral beams on the 8th of April, 1911, I found that

$$t=-150^\circ \text{ if } A=0.9 \text{ and } B=0.1,$$
$$t=-211^\circ \text{ if } A=0.99 \text{ and } B=0.01;$$

that is to say, *the temperature of the air above* 100 *kilometers was on that occasion probably between* -150° *and* -200° *C.*

I also calculated, as an *example of computation*,* the laws of absorption corresponding to the assumptions that

$t=-23^\circ$ at altitudes of less than 10 kilometers,
$t=-55^\circ$ at altitudes between 10 and 70 kilometers,
$t=-175^\circ$ at altitudes of more than 70 kilometers;

and taking the composition of the air at sea-level used by Wegener† as my starting point, I then obtained the following diagram:

* For more exact calculations we have probably to suppose the change of temperature at 70-kilometer level less sudden.

† Namely, hydrogen, 0.0033 per cent; helium, 0.0005; nitrogen, 78.1; oxygen, 20.9; argon, 0.937. We have not taken into account the existence of the hypothetical gas "geocoronium," supposed by Wegener.

From this it will be seen that the absorption at first increases slowly down to about 80 kilometers, corresponding to the long faint auroral beams. It then increases very rapidly, which may explain the bright lower edge of the auroral curtains such as those seen, for instance, in the photograph of an auroral curtain taken in Christiania on the 8th of April, 1911, by my assistant from my aurora expedition, the meteorologist Bernt Johannes Birkeland (see Plate XXVIII).

In my Geneva paper of 1907 I made the computations corresponding both to negative corpuscles like cathode rays and β-rays, and to positively charged corpuscles like α-rays. This latter hypothesis has been worked out by Mr. Vegard,* who has given a series of arguments for the view that aurora is due to α-rays. The well-defined straight beams in particular, and the sharp lower edge of the curtain, are in his opinion a decisive argument in favor of the α-ray hypothesis.

It would be very interesting to work out the absorption formulæ in this case, and compare them with those of negatively charged corpuscles.

In both cases the parallactic photographs of aurora, accompanied by photographs of the spectrum, seem able to give fundamental information about the upper air.

CARL STØRMER.

* See *Philosophical Magazine* for February, 1912.

THE GENERALIZATION OF ANALYTIC FUNCTIONS

ON THE THEORY OF WAVES AND GREEN'S METHOD *

THE GENERALIZATION OF ANALYTIC FUNCTIONS†

INTRODUCTION

THE generalization which is treated in the following pages has already been the subject of several investigations of mine, in the first place in several notes, published in the "Rendiconti" of the Reale Accademia dei Lincei, then in an extended memoir which appeared in the "Acta Matematica." Several of the lectures which I read at Stockholm were also devoted to this subject. And it is now my purpose, in returning to it, to consider the general case in some detail, beginning with the first foundations. In treating the general case it is necessary to consider certain elements, which I have called functions of hyperspaces, and which represent extensions of the functions of curves that I have already treated several times, in particular, in a recent course at the Sorbonne.

A space of n dimensions contains spaces of $0, 1, 2, \cdots n-1$ dimensions, and for that reason we consider functions of

*Three lectures presented at the inauguration of the Rice Institute, by Senator Vito Volterra, Professor of Mathematical Physics and Celestial Mechanics in the University of Rome.

†Translated from the Italian by Professor Griffith Conrad Evans, of the Rice Institute.

Vito Volterra

these spaces. We shall begin by extending to these functions the fundamental concepts of continuity and differentiation, and we shall consider the condition that a function be of the first degree. This condition depends upon an extension of Stokes's theorem. We shall then consider a relation between these functions analogous to that of monogeneity, which for functions in the ordinary sense was established by Cauchy. This leads to new types of equations with functional derivatives, which present analogies with the equation of Laplace.

We can separate the functions with which we are dealing into elementary and otherwise. The former have interesting properties and applications. A certain operation of composition turns out to possess quite curious arithmetical properties.

We shall finally develop the operations of differentiation and integration, and the extension of Cauchy's theorem in complete generality.

THE GENERALIZATION OF ANALYTIC FUNCTIONS

First Lecture

GENERAL OBSERVATIONS ON HYPERSPACES — GENERAL FORMULÆ FOR MATRICES, AND RELATIONS BETWEEN THE DIRECTION COSINES OF HYPERSPACES — FUNCTIONS OF HYPERSPACES AND THEIR DERIVATIVES — EXTENSION OF STOKES'S THEOREM — CONDITIONS WHICH THE DERIVATIVES OF FUNCTIONS OF HYPERSPACES MUST SATISFY, AND FORMULÆ FOR THE TRANSFORMATION OF COÖRDINATES — ISOGENEITY — CONDITIONS FOR ISOGENEITY.

1. *General observations on hyperspaces*

1. A hyperspace (space of n dimensions) will be characterized by the multiplicity of values of n independent variables $x_1, x_2, \cdots x_n$. A hyperspace S_r of r dimensions ($r < n$), contained in it, will correspond to the multiplicity of values which the $x_1, x_2, \cdots x_n$ assume when they are constrained

by $n - r$ independent relations, or in other words, when they depend on r independent variables $\omega_1, \omega_2, \cdots \omega_r$ to which they are bound by the n relations

$$(1) \qquad \begin{cases} x_1 = x_1(\omega_1, \omega_2, \cdots \omega_n) \\ x_2 = x_2(\omega_1, \omega_2, \cdots \omega_r) \\ \\ x_n = x_n(\omega_1, \omega_2, \cdots \omega_r) \end{cases}$$

We assume the differentiability of the preceding relations, and obtain

$$(2) \qquad dx_i = \sum_1^r{}_s \frac{\partial x_i}{\partial \omega_s} d\omega_s \qquad (i = 1, 2, \cdots n).$$

2. Let us consider the matrix

$$(3) \qquad \left\| \begin{matrix} \frac{\partial x_1}{\partial \omega_1} & \frac{\partial x_2}{\partial \omega_1} & \cdots & \frac{\partial x_n}{\partial \omega_1} \\ \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot \\ \frac{\partial x_1}{\partial \omega_r} & \frac{\partial x_2}{\partial \omega_r} & \cdots & \frac{\partial x_n}{\partial \omega_r} \end{matrix} \right\|$$

Let Δ^2 be the square of this matrix, and let us assume that if the sign of Δ is given at one point, it is fixed by continuity at all other points. When the sign of Δ is given we shall say that the *direction* of the hyperspace S_r is fixed. The quantity

$$dS_r = \Delta d\omega_1 d\omega_2 \cdots d\omega_r$$

will be called the *element of the hyperspace.*

Let us take a minor determinant of the matrix (3)

$$\Delta_{i_1 i_2 \cdots i_r} = \left| \begin{matrix} \frac{\partial x_{i_1}}{\partial \omega_1} & \frac{\partial x_{i_2}}{\partial \omega_1} & \cdots & \frac{\partial x_{i_r}}{\partial \omega_1} \\ \cdot & \cdot & \cdot & \cdot \\ \frac{\partial x_{i_1}}{\partial \omega_r} & \frac{\partial x_{i_2}}{\partial \omega_r} & \cdots & \frac{\partial x_{i_r}}{\partial \omega_r} \end{matrix} \right|$$

and write

$$\alpha_{i_1 i_2 \cdots i_r} = \frac{\Delta_{i_1 i_2 \cdots i_r}}{\Delta}. \tag{4}$$

The $\alpha_{i_1 i_2 \cdots i_r}$ will not change if we substitute for the ω_1, ω_2, $\cdots$ ω_r other variables bound by arbitrary relations to the first, and their signs will change only if we change the sign of the hyperspace; we shall call them the *direction cosines* of the hyperspace. We see at once that they must satisfy the relation

$$\Sigma_i \alpha^2_{i_1 i_2 \cdots i_r} = 1, \tag{A}$$

in which Σ_i denotes summation extended over all the combinations of the indices i_1, i_2, i_n.

3. If a space S_{n-r} has a point in common with S_r, and the direction cosines of S_{n-r} are denoted by $\beta_{h_1 \cdots h_{n-r}}$, we shall say that the two hyperspaces are normal to each other when we have the relation

$$\alpha_{i_1 i_2 \cdots i_r} = \beta_{h_1 h_2 \cdots h_{n-r}},$$

where all the i's are different from the h's, and the series of numbers $i_1 i_2, \cdots i_r, h_1, h_2, \cdots h_{n-r}$ is a permutation of the numbers 1, 2, $\cdots$ n, which is always odd or always even.

4. Whatever l may be, we can write

$$d\omega_l = \Sigma_i A_{i_1 i_2 \cdots i_{r-1}} \frac{d(x_{i_1}, x_{i_2} \cdots x_{i_{r-1}})}{d(\omega_1, \omega_2 \cdots \omega_{l-1} \omega_{l+1} \cdots \omega_r)} \quad (l = 1, 2, \cdots r) \tag{5}$$

in which the sum is extended over all the combinations of the indices $i_1, i_2 \cdots i_{r-1}$, and the A's are certain, in part indeterminate, infinitesimal parameters. In fact if we form the matrix of the coefficients of the A's, among its minors will be found the $r-1$th powers of the minors of the matrix (3), and so not all the minors of that matrix can be zero. If we substitute the values (5) in

the equations (2) we obtain

$$dx_s = -\Sigma_t \frac{d(x_s, x_{t_1} \cdots x_{t_{r-1}})}{d(\omega_1, \omega_2 \cdots \omega_r)} A_{t_1 t_2 \cdots t_{r-1}}. \tag{6}$$

Hence if $a_{t_1, t_2 \cdots t_{r-1}} = -\Delta A_{t_1 t_2 \cdots i_{r-1}}$ we shall have

$$dx_s = \Sigma_t a_{t_1 t_2 \cdots t_{r-1}} \alpha_{s t_1 t_2 \cdots t_{r-1}}. \tag{7}$$

5. Besides the equations (A) the α satisfy other relations, which we shall find in the next section.

2. *General formulæ about matrices. Relations between the direction cosines of a hyperspace*

1. We shall establish in this section several fundamental formulæ regarding the minors of matrices, which we shall often have occasion to use. Let us consider the two matrices

$$(1) \quad \left\| \begin{matrix} a_{11} & a_{12} \cdots a_{1n} \\ a_{21} & a_{22} \cdots a_{2n} \\ \cdot & \cdot \cdot \cdot \cdot \\ a_{r1} & a_{r2} \cdots a_{rn} \end{matrix} \right\| \qquad (2) \quad \left\| \begin{matrix} a_{11} & a_{12} \cdots a_{1n} \\ a_{21} & a_{22} \cdots a_{2n} \\ \cdot & \cdot \cdot \cdot \cdot \\ a_{p1} & a_{p2} \cdots a_{pn} \end{matrix} \right\|$$

the first with r rows, and the second with p rows, $(n > r \geq p)$, both however with the same elements. Let us write

$$\begin{vmatrix} a_{1t_1} & a_{1t_2} \cdots a_{1t_r} \\ \cdot & \cdot \cdot \cdot \cdot \\ a_{rt_1} & a_{rt_2} \cdots a_{rt_r} \end{vmatrix} = A_{t_1 t_2 \cdots t_r}, \qquad \begin{vmatrix} a_{1h_1} & a_{1h_2} \cdots a_{1h_p} \\ \cdot & \cdot \cdot \cdot \cdot \\ a_{ph_1} & a_{ph_2} \cdots a_{ph_p} \end{vmatrix} = B_{h_1 h_2 \cdots h_p}$$

and consider

$$\Delta_s = \begin{vmatrix} a_{st_1} & a_{st_1} \cdots a_{st_{r+1}} \\ a_{1t_1} & a_{1t_2} \cdots a_{1t_{r+1}} \\ \cdot & \cdot \cdot \cdot \cdot \\ a_{rt_1} & a_{rt_2} \cdots a_{rt_{r+1}} \end{vmatrix} = 0 \qquad (s = 1, 2, \cdots p).$$

We shall have

$$0 = \sum_{1\ s}^{p} \frac{\partial B_{h_1 h_2 \cdots h_p}}{\partial a_{sh_1}} \Delta_s$$

$$= -\sum_{1}^{p} \frac{\partial B_{h_1 h_2 \cdots h_p}}{\partial a_{sh_1}} \sum_{1\ t}^{r+1} (-1)^t a_{st_t} A_{t_1 \cdots t_{t-1} t_{t+1} \cdots t_{r+1}}$$

$$= -\sum_{1\ t}^{r+1} A_{t_1 \cdots t_{t-1} t_{t+1} \cdots t_{r+1}} \sum_{1}^{p} (-1)^s a_{st_t} \frac{\partial B_{h_1 \cdots h^{p}}}{\partial a_{sh_1}}.$$

From this it follows that

$$(3) \qquad \sum_{1\ t}^{r+1} (-1)^t A_{t_1 \cdots t_{t-1} t_{t+1} \cdots t_{r+1}} B_{t_t h_2 \cdots h_p} = 0.$$

2. This is the formula which we wished to obtain. In particular, if we take as identical the two matrices (1) and (2), we shall have

$$(3') \qquad \sum_{1\ t}^{r+1} (-1)^t A_{t_1 \cdots t_{t-1} t_{t+1} \cdots t_{r+1}} A_{t_t h_2 \cdots h_r} = 0.*$$

Among these equations let us notice specially the following, from which the others all follow:

$$(4) \qquad 0 = A_{t_1 t_2 h_1 \cdots h_{r-2}} A_{t_3 t_4 h_1 \cdots h_{r-2}} + A_{t_4 t_2 h_1 \cdots h_{r-2}} A_{t_1 t_3 h_1 \cdots h_{r-2}} + A_{t_1 t_4 h_1 \cdots h_{r-2}} A_{t_4 t_3 h_1 \cdots h_{r-2}}.\dagger$$

3. From the preceding formulæ we see that the direction cosines of a hyperspace must satisfy the relations

$$(B) \qquad \sum_{1\ s}^{r+1} (-1)^s \alpha_{t_1 t_2 \cdots t_{s-1} t_{s+1} \cdots t_{r+1}} \alpha_{t_s h_2 \cdots h_r} = 0.$$

* Vedi Antonelli: "Nota sulle relazioni indipendenti" ecc. ("Ann. d. Scuola Normale sup. di Pisa," Vol. III, page 71 e seg.)

† *Ibid.*, page 73.

3. *Functions of hyperspaces and their derivatives**

1. A variable ϕ will be said to be a *function of the hyperspace* S_r (of r dimensions) or a function of *order* r, if to every possible hyperspace with fixed direction corresponds a value of ϕ. This correspondence will be denoted by means of the symbol $\phi = \phi \mid [S_r] \mid$. We shall assume that we are dealing only with closed hyperspaces S_r.†

Let us take a point P of S_r and through it draw a hyperspace S_{n-r} normal to S_r, taking in S_{n-r} a small neighborhood s of P. If we make P describe all the points of S_r we shall generate a portion of n-dimensional space, which we shall call a neighborhood of S_r. While P is describing S_r any other point P' of s describes a new hyperspace S_r', which we shall say belongs to the neighborhood of S_r. The function $\phi \mid [S_r] \mid$ will be said to be continuous if, when we take a quantity σ arbitrarily small, we can find a neighborhood of S_r such that

$$\text{mod } [\phi \mid [s_r'] \mid - \phi \mid [s_r] \mid] < \sigma,$$

where S_r' belongs to that neighborhood.

Besides the continuity of $\phi \mid [S_r] \mid$ let us admit also the following property. Let us pass from the hyperspace S_r to the hyperspace S_r' by giving to each point of S_r a displacement ϵ which varies continuously from point to point. The displacement ϵ generates a hyperspace S_{r+1} of $r+1$ dimensions, of amplitude say, σ. We shall assume that we can make $\{\phi \mid [S_r] \mid - \phi \mid [S_r] \mid\}$ less than a number chosen arbitrarily small, provided σ be less than some value σ_0.

2. With this understood, take in S_r a neighborhood s of a point P, and give to s a displacement δx_i parallel to x_i.

* Vedi la mia Nota I: "Sulle funzioni dipendenti da linee." ("Atti d. R. Acc d. Lincei," Vol. III, fasc. 9.)

† Vedi: Betti: "Sopra gli spazii di un numero qualunque di dimensioni." (Annali di Mat., T. IV.)

Let us denote by $\delta\phi$ the corresponding variation of ϕ, and let us suppose that the value

$$\lim_{\substack{s=0\\ \delta x_i=0}} \frac{\partial\phi}{s\cdot\partial x_i} = \phi'_{x_i} \qquad (i = 1, 2, \cdots n)$$

exists. We shall call this *the derivative of ϕ with respect to x_i at the point P.* With the assumption that the ratio which appears in the left-hand member approaches its limit uniformly, with respect to all possible points P and hyperspaces S_r, and that this limit is continuous, we can easily verify the fact that if we give to every point of S_r a displacement of components $\delta x_1, \delta x_2, \cdots Sx_n$, the corresponding variation of ϕ is given, except for infinitesimals of higher order, by the formula

$$\delta\phi = \int_{S_r} \sum_1^n{}_i \phi'_{x_i}\delta x_i dS_r. \tag{1}$$

3. Let us find out now what conditions the ϕ'_{x_i} must satisfy. If the displacements are such as to carry the space S_r into itself, the quantity $\delta\phi$ must vanish. Hence we must have $\delta\phi = 0$ if we take (see § 1, form 7)

$$\delta x_i = \sum_h a_{h_1 h_2 \cdots h_{r-1}} \alpha_{ih_1 \cdots h_{r-1}}$$

whatever the quantities a may be. Hence

$$0 = \int_{S_r} \sum_h{}_{h_1 h_2 \cdots h_{r-1}} \sum_1^n{}_i \phi'_{x_i} \alpha_{ih_1 \cdots h_{r-1}} dS_r$$

and from this we have

$$\sum_1^n{}_i \phi'_{x_i} \alpha_{ih_1 \cdots h_{r-1}} = 0 \tag{2}$$

for every possible combination of the indices $h_2 \cdots h_{r-1}$.

4. Since now the α satisfy the relations § 2, (B), we have

$$\sum_1^{r+1}{}_i (-1)^i \alpha_{q_i h_1 \cdots h_{r-1}} \alpha_{q_1 \cdots q_{i-1} q_{i+1} \cdots q_{r+1}} = 0.$$

If we multiply this by an undetermined parameter $\lambda_{q_1 q_2 \cdots q_{r+1}}$ which satisfies the condition that it changes sign for every transposition of the indices, we shall have

$$0 = \sum_q \lambda_{q_1 q_2 \cdots q_{r+1}} \sum_1^{r+1} {}_t (-1)^t \alpha_{q_t h_1 \cdots h_{r-1}} \alpha_{q_1 \cdots q_{t-1} q_{t+1} \cdots q_{r+1}}$$

$$= \sum_1^n {}_t \sum_q \lambda_{t q_1 \cdots q_r} \alpha_{q_1 \cdots q_r} \alpha_{t h_1 \cdots h_{r-1}}$$

and subtracting this from equation (2),

$$0 = \sum_1^n {}_t \left\{ \phi'_{x_t} - \sum_q \lambda_{t q_1 \cdots q_r} \alpha_{q_1 q_2 \cdots q_r} \right\} \alpha_{t h_1 \cdots h_{r-1}}$$

whence

$$\phi'_{x_t} = \sum_q \lambda_{t q_1 \cdots q_r} \alpha_{q_1 q_2 \cdots q_r} \tag{3}$$ *

From this it follows that

$$\delta\phi = \int_{S_r} \sum_1^n {}_t \sum_q \lambda_{t q_1 \cdots q_r} \alpha_{q_1 q_2 \cdots q_r} \delta x_t \, dS_r$$

$$= \int_{S_r} \sum_q \lambda_{q_1 q_2 \cdots q_{r+1}} \left\{ \sum_1^{r+1} {}_t (-1)^{t-1} \alpha_{q_1 q_2 \cdots q_{t-1} q_{t+1} \cdots q_{r+1}} \delta x_{q_t} \right\} dS_r.$$

Consider now the elements dS_r and suppose drawn through every point of it a segment of components $\delta x_1 \cdots \delta x_n$. The locus of these segments will be a space S_{r+1} of $r+1$ dimensions. If the equations of the hyperspace S_r are

$$x_t = x_t(\omega_1, \omega_2, \cdots \omega_r) \qquad (t = 1, 2, \cdots n)$$

the equations of the hyperspace S_{r+1} will be

$$x_t = x_t(\omega_1, \omega_2, \cdots \omega_r) + \omega_{r+1} \delta x_t \qquad (t = 1, 2, \cdots n).$$

* See my Note II: "Sulle funzioni dipendenti da linee." ("Atti della R. Acc. dei Lincei," Vol. III, fasc. 10.)

Let us form the matrix

$$(4) \qquad \left\| \begin{matrix} \frac{\partial x_1}{\partial \omega_1}, & \frac{\partial x_2}{\partial \omega_1} \cdots \frac{\partial x_n}{\partial \omega_1} \\ \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \\ \frac{\partial x_1}{\partial \omega_r}, & \frac{\partial x_2}{\partial \omega_r} \cdots \frac{\partial x_n}{\partial \omega_r} \\ \partial x_1, & \partial x_2 \cdots \partial x_n \end{matrix} \right\|$$

Let us denote its square by Δ^2_{r+1}, and the square of the matrix obtained from it by taking away the last line by Δ^2_r. We shall have

$$\Delta^2_{r+1} = \Delta^2_r \left\{ \sum_q \sum_1^{r+1} {}_t (-1)^{t-1} \alpha_{q_1 \cdots q_{t-1} q_{t+1} \cdots q_{r+1}} \delta x_{q_t} \right\}^2$$

We can fix the direction of S_{r+1} with respect to S_r in such a way that

$$\Delta_{r+1} = (-1)^r \Delta_r \sqrt{\sum_q \sum_1^{r+1} {}_t (-1)^{t-1} \alpha_{q_1 \cdots q_{t-1} q_{t+1} \cdots q_{r+1}} \delta x_{q_t}}$$

where the sign of the radical is taken as positive. If now we denote the direction cosines of S_{r+1} by $\beta_{q_1 q_2 \cdots q_{r+1}}$, which are calculated from the matrix (4), we shall have finally

$$\delta\phi = \int_{S_{r+1}} \sum_q \lambda_{q_1 q_2 \cdots q_{r+1}} \beta_{q_1 q_2 \cdots q_{r+1}} dS_{r+1}.$$

Hence if S_r is a movable hyperspace which passes from S'_r to S''_r, thus generating a S_{r+1}, we shall have

$$(5) \qquad \phi|[S''_r] - \phi|[S'_r]| = \int_{S_{r+1}} \sum_q \lambda_{q_1 q_2 \cdots q_{r+1}} \beta_{q_1 q_2 \cdots q_{r+1}} dS_{r+1}.$$

It is well to note explicitly that besides varying from point to point of the total hyperspace (of n dimensions), the parameters λ may also vary for one and the same point according to the hyperspace to which they refer, and even

for the same hyperspace one set of λ's may be substituted for another provided the relations (3) are always satisfied.

5. A function $\phi\,|[S_r]\,|$ will be said to be *regular* (or *simple*) when the following condition is satisfied. Let S_r' and S_r'' be two hyperspaces having a common portion s, whose direction is different according as it is considered as belonging to the first or the second hyperspace. Denote by S_r''' the hyperspace which we get by taking away s from the combination of S_r' and S_r''' and fix as its direction the direction of those two hyperspaces. We impose the condition

$$\phi\,|[S_r''']\,| = \phi\,|[S_r']\,| + \phi\,|[S_r'']|.$$

When ϕ is regular it follows immediately that if S_r decreases indefinitely in amplitude

$$\lim \phi\,|[S_r]| = 0. \tag{C}$$

We have then immediately the further property that if S_r and S_r' are two hyperspaces with a common point P, whose elements at P are contained in a single S_{r+1}, of $r+1$ dimensions,

$$\sum_q (\lambda_{q_1 q_2 \cdots q_{r+1}} - \lambda'_{q_1 q_2 \cdots q_{r+1}})\,\beta_{q_1 q_2 \cdots q_{r+1}} = 0 \tag{6}$$

where λ and λ' are the parameters which correspond to $\phi\,|[S_r]|$ and $\phi\,|[S_r']|$ at the point P, and the β's are the direction cosines of S_{r+1}.

Upon this basis let us consider a hyperspace S_r passing through the point P, whose element at P is defined by the equations

$$dx_i = \sum_1^r{}_s\, a_{is}\, d\omega_s \qquad (i = 1, 2, \cdots n)$$

and let $S_r^{(t_1 \cdots t_r)(h_1 \cdots h_p)}$ denote hyperspaces passing through P defined by the equations

$$dx_{t_s} = a_{t_s, s}\, d\omega_s + \sum_1^p{}_t\, a_{t_s, h_t}\, d\omega_{h_t} \qquad \begin{cases} s = 1, 2, \cdots r \\ s \neq h_1, h_2 \cdots h_p. \end{cases}$$

$$dx_{t_v} = \sum_1^p{}_t\, a_{t_v, h_t}\, d\omega_{ht} \qquad (v = h_1, h_2 \cdots h_p, r+1, \cdots n.)$$

In particular let us consider the hyperspaces $S^{(i_1 \cdots i_{s-1} i_{s+1} \cdots i_{r+1})}$ and $S_r^{(i_1 \cdots i_{t-1} i_{t+1} \cdots i_{r+1})}$ whose elements at P are contained in a hyperspace of $r+1$ dimensions, of which the direction cosines β are zero, except $\beta_{i_1 i_2 \cdots i_{r+1}} = 1$. By means of (6), we have

$$\lambda^{(i_1 \cdots i_{s-1} i_{s+1} \cdots i_{r+1})}_{i_1 i_2 \cdots i_{r+1}} = \lambda^{(i_1 \cdots i_{t-1} i_{t+1} \cdots i_{r+1})}_{i_1 i_2 \cdots i_{r+1}}$$

where the indices $i_1 i_2 \cdots i_r$ denote the parameters λ corresponding to the hyperspace $S_r^{(i_1 \cdots i_n)}$. Therefore we can suppress the indices and write simply

$$\lambda^{(i_1 \cdots i_{s-1} i_{s+1} \cdots i_{r+1})}_{i_1 i_2 \cdots i_{r+1}} = \Lambda_{i_1 i_2 \cdots i_{r+1}}. \tag{7}$$

6. Two hyperspaces $S_r^{(i_1 \cdots i_r)(h_1 \cdots h_{p-1})}$ and $S_r^{(i_1 \cdots i_r)(h_1 \cdots h_p)}$ have elements at P which are contained in a S_{r+1}, whose element at P is defined by the equations

$$dx_{i_s} = a_{i_s, s} d\omega_s + \sum_1^p{}_t a_{i_s, h_t} d\omega_{h_t} \qquad \begin{cases} s = 1, 2, \cdots r \\ s \neq h_1, h_2, \cdots h_p \end{cases}$$

$$dx_{i_{h_p}} = a_{i_{h_p}, h_p} d\omega_{r+1} + \sum_1^p{}_t a_{i_{h_p}, h_t} d\omega_{h_t}$$

$$dx_{i_v} = \sum_1^p{}_t a_{i_v, h_t} d\omega_{h_t}, \qquad (v = h_1, h_2 \cdots h^{p-1}, r+1, \cdots n.)$$

Hence, if we denote by β the direction cosines of S_{r+1} and by $\alpha^{(i_1 \cdots i_r)(h_1 \cdots h_p)}$ the direction cosines of $S_r^{(i_1 \cdots i_r)(h_1 \cdots h_p)}$, we shall have

$$\frac{\beta_{i_{h_p} m_1 m_2 \cdots m_r}}{\alpha^{(i_1 \cdots i_r)(h_1 \cdots h_p)}_{m_1 m_2 \cdots m_r}} = \kappa,$$

where κ is independent of the indices $m_1, m_2, \cdots m_r$, and all the β's are zero, in the indices of which i_{h_p} is missing. From this it follows by reason of (6) that

$$\sum{}_m \left(\lambda^{(i_1 \cdots i_r)(h_1 \cdots h_p)}_{i_{h_p} m_1 \cdots m_r} - \lambda^{(i_1 \cdots i_r)(h_1 \cdots h_{p-1})}_{i_{h_p} m_1 \cdots m_r} \right) \alpha^{(i_1 \cdots i_r)(h_1 \cdots h_p)}_{m_1 \cdots m_r} = 0,$$

where the index $(i_1 \cdots i_r)(h_1 \cdots h_p)$, affixed to the λ, means that refers to the hyperspace having the same index. We have then

$$\sum_m \lambda_{t_{h_p} m_1 \cdots m_r}^{(t_1 \cdots t_r)(h_1 \cdots h_p)} \alpha_{m_1 \cdots m_r}^{(t_1 \cdots t_r)(h_1 \cdots h_p)} = \sum_m \lambda_{t_{h_p} m_1 \cdots m_r}^{(t_1 \cdots t_r)(h_1 \cdots h_{p-1})} \alpha_{m_1 \cdots m_r}^{(t_1 \cdots t_r)(h_1 \cdots h_p)},$$

in which, by means of (3), we can substitute for the $\lambda_{t_{h_p} m_1 \cdots m_r}^{(t_1 \cdots t_r)(h_1 \cdots h_p)}$ the $\lambda_{t_{h_p} m_1 \cdots m_r}^{(t_1 \cdots t_r)(h_1 \cdots h_{p-1})}$, and consequently, the $\Lambda_{t_{h_p} m_1 \cdots m_r}$ of formula (7).

We observe however that the hyperspace $S_r^{(t_1 \cdots t_r)(t_1 \cdots t_p)}$ is nothing but the hyperspace S_r, and therefore we can take for the λ's belonging to this space, at the point P, the λ's without index of formula (6). We have then the theorem

If ϕ is a regular function of the hyperspace S_r, contained in a hyperspace S_n, there exist for every point of S_n a system of values which can be considered as the parameters $\lambda_{i_1 i_2 \cdots i_{r+1}}$ for all the hyperspaces S_r which pass through that point.

7. From the equations (5) (C), assuming that $\phi\,|\,[S_r]\,|$ is regular we get,

$$(5') \qquad \phi\,|\,[S_r]\,| = \int_{S_r} {}_1 \sum_q \Lambda_{q_1 q_2 \cdots q_{r+1}} \beta_{q_1 q_2 \cdots q_{r+1}} dS_{r+1}.$$

Here S_{r+1} is an arbitrary hyperspace of $r+1$ dimensions, whose boundary is S_r. If S_{r+1} grows indefinitely smaller about a point P, by writing

$$S_{r+1} = \int_{S_{r+1}} dS_{r+1}$$

we shall have

$$\lim \frac{\phi\,|\,[S_r]\,|}{S_{r+1}} = \sum_q \Lambda_{q_1 q_2 \cdots q_{r+1}} \beta_{q_1 q_2 \cdots q_{r+1}} = \frac{d\phi}{dS_{r+1}}$$

where the β are the direction cosines of S_{r+1} at P.

Let us take $S_{r+1} = S_{r+1}^{(i_1 i_2 \cdots i_{r+1})}$ such that at P all the direction cosines β shall be zero except $\beta_{i_1 i_2 \cdots i_{r+1}} = 1$. We shall have

$$\lim \frac{\phi\,|\,[S_r]\,|}{S_{r+1}^{(i_1 i_2 \cdots i_{r+1})}} = \Lambda_{i_1 i_2 \cdots i_{r+1}}.$$

Therefore we shall write

$$\Lambda_{i_1 i_2 \cdots i_{r+1}} = \frac{\partial \phi}{\partial (x_{i_1} x_{i_2} \cdots x_{i_{r+1}})},$$

and define this quantity as the *derivative of ϕ with respect to* $x_{i_1} x_{i_2} \cdots x_{i_{r+1}}$. What relations must these derivatives satisfy? Before proceeding to the search for these relations, it will be necessary to give an extension of Stokes's theorem, a subject which is dealt with in the next section.

4. *Extension of Stokes's theorem*

1. Let $L_{i_1 i_2 \cdots i_r}$ be functions of the points of the hyperspace S_n, such that every transposition of the indices creates a change of sign, and form the expression

$$(1) \qquad M_{i_1 i_2 \cdots i_{r+1}} = \sum_1^{r+1} {}_s (-1)^{s-1} \frac{\partial L_{i_1 \cdots i_{s-1} i_{s+1} \cdots i_{r+1}}}{\partial x_{i_s}}.$$

Let S_{r+1} be a hyperspace, of $r+1$ dimensions, bounded by a set of hyperspaces S_r, let its direction cosines be $\alpha_{i_1 i_2 \cdots i_{r+1}}$, and form the expression

$$\int_{S_{r+1}} \Omega dS_{r+1},$$

putting $$\Omega = \Sigma_i M_{i_1 i_2 \cdots i_{r+1}} \alpha_{i_1 i_2 \cdots i_{r+1}}.$$

If the equations of S_{r+1} are

$$x_i = x_i(\omega_1, \omega_2 \cdots \omega_{r+1}) \qquad (i = 1, 2, \cdots n),$$

we shall have

$$\Omega dS_{r+1}$$

$$= \sum_t M_{t_1 t_2 \cdots t_{r+1}} \frac{d(x_{t_1} x_{t_2} \cdots x_{t_{r+1}})}{d(\omega_1 \omega_2 \cdots \omega_{r+1})} d\omega_1 d\omega_2 \cdots d\omega_{r+1}$$

$$= \sum_t \sum_1^n{}_s \frac{\partial L_{t_1 \cdots t_{s-1} t_{s+1} \cdots t_{r+1}}}{\partial x_{t_s}} \frac{d(x_{t_s} x_{t_1} \cdots x_{t_{s-1}} x_{t_{s+1}} \cdots x_{t_{r+1}})}{d(\omega_1 \omega_2 \cdots \omega_{r+1})} d\omega_1 d\omega_2 \cdots d\omega_{r+1}.$$

$$= \sum_t \frac{d(L_{t_1 \cdots t_r}, x_{t_1} x_{t_2} \cdots x_{t_r})}{d(\omega_1, \omega_2, \cdots \omega_{r+1})} d\omega_1 d\omega_2 \cdots d\omega_{r+1}$$

$$= \sum_t \sum_1^{r+1}{}_i (-1)^{i-1} \frac{\partial L_{t_1 \cdots t_r}}{\partial \omega_t} \frac{d(x_1 \cdots x_{t_r})}{d(\omega_1 \cdots \omega_{t-1} \omega_{t+1} \cdots \omega_{r-1})} d\omega_1 d\omega_2 \cdots d\omega_{r+1}.$$

Hence

$$\int_{S_{r+1}} \Omega dS_{r+1}$$

$$= \int_{S_r} \sum_t \sum_1^{r+1}{}_i L_{t_1 t_2 \cdots t_r} \frac{d(x_{t_1} x_{t_2} \cdots x_{t_r})}{d(\omega_1 \cdots \omega_{t-1} \omega_{t+1} \cdots \omega_{r+1})} d\omega_1 \cdots d\omega_{t-1} d\omega_{t+1} \cdots d\omega_{r+1}.$$

We can make the hyperspace S depend on r independent parameters $\bar{\omega}_1, \bar{\omega}_2 \cdots \bar{\omega}_r$, whence we shall have

$$\int_{S_{r+1}} \Omega dS_{r+1} = \int_{S_r} \sum_t L_{t_1 t_2 \cdots t_r} \frac{d(x_{t_1} x_{t_2} \cdots x_{t_r})}{d(\bar{\omega}_1 \bar{\omega}_2 \cdots \bar{\omega}_r)} d\bar{\omega}_1 d\bar{\omega}_2 \cdots d\bar{\omega}_r.$$

From this comes the formula

$$(2) \quad \int_{S_{r+1}} \sum_t M_{t_1 t_2 \cdots t_{r+1}} \alpha_{t_1 t_2 \cdots t_{r+1}} dS_{r+1} = \int_{S_r} \sum_t L_{t_1 t_2 \cdots t_r} \beta_{t_1 t_2 \cdots t_r} dS_r,$$

where the β's are the direction cosines of the hyperspace S_r.

2. From these formulæ it follows that if

$$\int_{S_r} \sum_t L_{t_1 t_2 \cdots t_r} \beta_{t_1 t_2 \cdots t_r} dS = 0,$$

for every closed hyperspace S_r in the region S_n, the necessary and sufficient conditions that must be satisfied are

$$(3) \qquad M_{i_1 i_2 \cdots i_{r+1}} = \sum_1^{r+1} {}_s (-1)^{s-1} \frac{\partial L_{i_1 \cdots i_{s-1} i_{s+1} \cdots i_{r+1}}}{\partial x_{i_s}} = 0$$

for every combination of the indices $i_1 i_2 \cdots i_{r+1}$.

5. *Conditions which the derivatives of functions of hyperspaces must satisfy. Formulæ for the change of coördinates*

1. Let $\phi\,|\,[S_r]\,|$ be regular, and return to formula (5′) of section 3. Since the integral which appears in the right-hand member does not change when S_{r+1} changes, provided the boundary S_r does not change, we must have

$$\int_{S_{r+1}} \sum\nolimits_q \Lambda_{q_1 q_2 \cdots q_{r+1}} \beta_{q_1 q_2 \cdots q_{r+1}} dS_{r+1} = 0$$

when the integration is extended over any closed hyperspace S_{r+1}. Hence the necessary and sufficient conditions which the Λ must satisfy in order to be the derivatives of a regular function of hyperspaces S_r (see section 4, article 2) is

$$(D) \qquad \sum_1^{r+2} {}_s (-1)^{s-1} \frac{\partial \Lambda_{i_1 \cdots i_{s-1} i_{s+1} \cdots i_{r+2}}}{\partial x_{i_s}} = 0$$

for every possible combination of the indices $i_1 i_2 \cdots i_{r+2}$. We can write these equations, making use of the symbols of section 3, article 7, in the form

$$(D') \qquad \sum_1^{r+2} {}_s (-1)^{s-1} \frac{\partial}{\partial x_{i_s}} \frac{\partial \phi}{\partial (x_{i_1} \cdots x_{i_{s-1}} x_{i_{s+1}} \cdots x_{i_{r+2}})} = 0.$$

We shall call these conditions the *conditions of integrability*.

2. Consider now the formulæ for change of variable, transforming the variables $x_1, x_2 \cdots x_n$ into $x_1', x_2' \cdots x_n'$ by

means of the relations

$$x_i' = x_i'(x_1, x_2, \cdots x_n) \qquad (i = 1, 2, \cdots n)$$

such that $$\frac{d(x_1', x_2' \cdots x_n')}{d(x_1, x_2 \cdots x_n)}$$

is always finite and different from zero. Let us consider two regions which correspond in a one-to-one manner, S_n and S_n', one belonging to the first set of variables, the other to the second. Let S_{r+1} be a hyperspace, bounded by S_r and contained in S_n, and let S_{r+1}', bounded by S_r', correspond to it in S_n'. If we suppose that S_{r+1} is given by the equations

$$x_i = x_i(\omega_1 \omega_2 \cdots \omega_{r+1}) \qquad (i = 1, 2 \cdots n),$$

we shall have

$$\phi\,|\,[S_r]\,|$$

$$= \int_{S_{r+1}} \sum_i \frac{\partial \phi}{\partial (x_{i_1} x_{i_2} \cdots x_{i_{r+1}})} \frac{d(x_{i_1} \cdots x_{i_{r+1}})}{d(\omega_1 \cdots \omega_{r+1})} d\omega_1 d\omega_2 \cdots d\omega_{r+1}$$

$$= \int_{S_{r+1}} \sum_i \frac{\partial \phi}{\partial (x_{i_1} \cdots x_{i_{r+1}})} \sum_h \frac{d(x_{i_1} \cdots x_{i_{r+1}})}{d(x_{h_1}' \cdots x_{h_{r+1}}')} \frac{d(x_{h_1}' \cdots x_{h_{r+1}}')}{d(\omega_1 \cdots \omega_{r+1})} d\omega_1 \cdots d\omega_{r+1}$$

$$= \int_{S_{r+1}} \sum_h \frac{d(x_{h_1}' \cdots x_{h_{r+1}}')}{d(\omega_1 \cdots \omega_{r+1})} \sum_i \frac{\partial \phi}{\partial (x_{i_1} \cdots x_{i_{r+1}})} \frac{d(x_{i_1} \cdots x_{i_{r+1}})}{d(x_{h_1}' \cdots x_{h_{r+1}}')} d\omega_1 \cdots d\omega_{r+1}$$

$$= \int_{S_{r+1}'} \sum_h \beta_{h_1 \cdots h_{r+1}}' \left(\sum_i \frac{\partial \phi}{\partial (x_{i_1} \cdots x_{i_{r+1}})} \frac{d(x_{i_1} \cdots x_{i_{r+1}})}{d(x_{h_1}' \cdots x_{h_{r+1}}')} \right) dS_{r+1}'$$

where the β' denote the direction cosines of S_{r+1}'.

If we write

$$\Lambda_{h_1 h_2 \cdots h_{r+1}}' = \sum_i \frac{\partial \phi}{\partial (x_{i_1} \cdots x_{i_{r+1}})} \frac{d(x_{i_1} \cdots x_{i_{r+1}})}{d(x_{h_1}' \cdots x_{h_{r+1}}')}$$

we shall have

$$\phi\,|\,[S_r]\,| = \phi\,|\,[S_r']\,| = \int_{S_{r+1}'} \sum_h \Lambda_{h_1 h_2 \cdots h_{r+1}}' \beta_{h_1 h_2 \cdots h_{r+1}}' dS_{r+1}',$$

whence $$\Lambda_{h_1 \cdots h_{r+1}}' = \frac{\partial \phi}{\partial (x_{h_1}' x_{h_2}' \cdots x_{h_{r+1}}')}.$$

The desired formulæ for the transformation of coördinates become then

$$(1)\qquad \frac{\partial\phi}{\partial(x'_{h_1}x'_{h_2}\cdots x'_{h_{r+1}})}=\sum_t \frac{\partial\phi}{\partial(x_{t_1}x_{t_2}\cdots x_{t_{r+1}})}\frac{d(x_{t_1}x_{t_2}\cdots x_{t_{r+1}})}{d(x'_{h_1}x'_{h_2}\cdots x'_{h_{r+1}})}.$$

3. If we multiply the preceding equations by

$$\frac{d(x_{s_{r+2}}x_{s_{r+3}}\cdots x_{s_n})}{d(x'_{h_{r+2}}x_{h_{r+3}}\cdots x_{h_n})}$$

and add them, for all values of the h's, we shall have

$$\sum_h \frac{\partial\phi}{\partial(x'_{h_1}x'_{h_2}\cdots x'_{h_{r+1}})}\frac{d(x_{s_{r+2}}x_{s_{r+3}}\cdots x_{s_n})}{d(x'_{h_{r+2}}x'_{h_{r+3}}\cdots x_{h_n})}$$

$$=\frac{\partial\phi}{\partial(x_{s_1}x_{s_2}\cdots x_{s_{r+1}})}\frac{d(x_1x_2\cdots x_n)}{d(x'_1x'_2\cdots x'_n)}$$

where

$$(h_1, h_2\cdots h_{r+1}, h_{r+2}\cdots h_n)\equiv(s_1, s_2\cdots s_{r+1}, s_{r+2}\cdots s_n)\equiv(1, 2, \cdots n),$$

the notation being used to denote the fact that the groups of the h's and of the s's are two even permutations of the first m integers. Hence

$$(2)\qquad \frac{\partial\phi}{\partial(x_{s_1}x_{s_2}\cdots x_{s_{r+1}})}$$

$$=\frac{1}{\dfrac{d(x_1x_2\cdots x_n)}{d(x'_1x'_2\cdots x'_n)}}\sum_h \frac{\partial\phi}{\partial(x'_{h_1}x'_{h_2}\cdots x'_{h_{r+1}})}\frac{d(x_{s_{r+2}}x_{s_{r+3}}\cdots x_{s_n})}{d(x'_{h_{r+2}}x'_{h_{r+3}}\cdots x'_{h_n})}.$$

4. By means of the equations (D′), which are satisfied by the functions $\dfrac{\partial\phi}{\partial(x_{s_1}\cdots x_{s_{r+1}})}$, and the analogous equations satisfied by the functions $\dfrac{\partial\phi}{\partial(x'_{h_1}\cdots x'_{h_{r+1}})}$, we obtain the theorem:

If the quantities $a_{i_1 i_2 \cdots i_{r+1}}$ *(which change sign for every transposition in the indices) satisfy the equations*

$$(3) \qquad \sum_1^{r+2}{}_s (-1)^{s-1} \frac{\partial a_{i_1 \cdots i_{s-1} i_{s+1} \cdots i_{r+2}}}{\partial x_{i_s}} = 0$$

the quantities $a'_{h_1 h_2 \cdots h_{r+1}}$ *given by the formulæ*

$$a'_{h_1 h_2 \cdots h_{r+1}} = \frac{1}{\dfrac{d(x'_1 \cdots x'_n)}{d(x_1 \cdots x_n)}} \sum_i a_{i_1 i_2 \cdots i_{r+1}} \frac{d(x'_{h_{r+2}} x'_{h_{r+3}} \cdots x'_{h_n})}{d(x_{i_{r+2}} x_{i_{r+3}} \cdots x_{i_n})}$$

$$(i_1, i_2 \cdots i_n) \equiv (h_1, h_2 \cdots h_n) \equiv (1, 2, \cdots n)$$

will satisfy the analogous equations

$$(3') \qquad \sum_1^{r+2}{}_s (-1)^{s-1} \frac{\partial a'_{h_1 \cdots h_{s-1} h_{s+1} \cdots h_{r+2}}}{\partial x'_{h_s}} = 0.$$

5. Let us write $\dfrac{\partial \phi}{\partial (x_{i_1} \cdots x_{i_{r+1}})} = a_{i_1 \cdots i_{r+1}}$.

We wish to show that if the following conditions are satisfied

$$(4) \qquad \sum_1^{r+2} (-1)^s a_{i_1 \cdots i_{s-1} i_{s+1} \cdots i_{r+2}} a_{i_s h_1 \cdots h_r} = 0$$

and we make a change of variables from the $x_1, x_2 \cdots x_n$ to the $x'_1, x'_2, \cdots x'_n$, we shall obtain the result that the quantities

$$a'_{h_1 \cdots h_{r+1}} = \frac{\partial \phi}{\partial (x'_{h_1} \cdots x'_{hr+1})}$$

will satisfy the analogous equations

$$(4') \qquad \sum_1^{r+2}{}_s (-1)^s a'_{h_1 \cdots h_{s-1} h_{s+1} \cdots h_{r+2}} a'_{h_s i_1 \cdots i_r} = 0.$$

In fact if we have the relations (4), the $a_{i_1 \cdots i_{r+1}}$ will be minor determinants of a matrix

$$\begin{Vmatrix} A_{1,1} & A_{1,2} \cdots A_{1,n} \\ \cdot & \cdot \cdot \cdot \cdot \cdot \cdot \\ A_{r+1,1} & A_{r+1,2} \cdots A_{r+1,n} \end{Vmatrix},$$

that is, we can write

$$a_{i_1 \cdots i_{r+1}} = \begin{vmatrix} A_{1,i_1} & \cdots & A_{1,i_{r+1}} \\ \cdot & \cdot \cdot \cdot \cdot \cdot \\ A_{r+1,i_1} & \cdots & A_{r+1,i_{r+1}} \end{vmatrix}.$$

If we write

$$\frac{\partial x_s}{\partial x'_i} = B_{is}$$

we shall have, by means of (1), the equations

$$a'_{h_1 \cdots h_{r+1}} = \sum_{i} \begin{vmatrix} A_{1,i_1} \cdots A_{1,i_{r+1}} \\ \cdot \cdot \cdot \cdot \cdot \\ A_{r+1,i_1} \cdots A_{r+1,i_{r+1}} \end{vmatrix} \begin{vmatrix} B_{h_1 i_1} \cdots B_{h_1, i_{r+1}} \\ \cdot \cdot \cdot \cdot \cdot \\ B_{h_{r+1}, i_1} \cdots B_{h_{r+1}, i_{r+1}} \end{vmatrix}$$

that is, if we define $\sum_{1}^{\infty} {}_s A_{is} B_{hs} = C_{ih}$, the relations

$$a'_{h_1 \cdots h_{r+1}} = \begin{vmatrix} C_{1,h_1} \cdots C_{1,h_{r+1}} \\ C_{r+1,h_1} \cdots C_{r+1,h_{r+1}} \end{vmatrix}.$$

In other words, the quantities $a'_{h_1 \cdots h_{r+1}}$ are minor determinants of the matrix

$$\begin{Vmatrix} C_{11} & C_{12} \cdots C_{1n} \\ C_{21} & C_{22} \cdots C_{2n} \\ \cdot & \cdot \cdot \cdot \cdot \cdot \cdot \\ C_{r+1,1} & C_{r+1,2} \cdots C_{r+1,n} \end{Vmatrix}$$

and so the equations (4′) will be satisfied.

When the equations (4) are satisfied, the function $\phi|[S_r]|$ is said to be *elementary* (see §§ 10, 14).

6. *Isogeneity* *

1. Two complex functions f, ϕ, of hyperspaces S_r, which are *regular*, are said to be *isogenous* if in every point of the total hyperspace S_n, the ratio

$$\frac{\dfrac{d\phi}{dS_{r+1}}}{\dfrac{df}{dS_{r+1}}}$$

is independent of the hyperspace S.

Separating the real and imaginary parts, let us write

$$\frac{\partial f}{\partial(x_{i_1}x_{i_2}\cdots x_{i_{r+1}})} = p_{i_1\cdots i_{r+1}} + iq_{i_1\cdots i_{r+1}} = p_I + iq_I$$

$$\frac{\partial \phi}{\partial(x_{i_1}x_{i_2}\cdots x_{i_{r+1}})} = \omega_{i_1\cdots i_{r+1}} + i\chi_{i_1\cdots i_{r+1}} = \overline{\omega}_I + i\chi_I$$

where I denotes the set of indices $i_1 i_2 \cdots i_{r+1}$, that is, $I \equiv (i_1 \cdots i_{r+1})$. The necessary and sufficient condition in order that f and ϕ be isogenous may be written

$$\frac{\overline{\omega}_I + \chi_I}{p_I + q_I} = \frac{\overline{\omega}_H + i\chi_H}{p_H + iq_H}, \tag{1}$$

where $H \equiv (h_1 h_2 \cdots h_{r+1})$ is another arbitrary combination of the indices. From the preceding equations we find

$$\begin{cases} \overline{\omega}_I p_H - \overline{\omega}_H p_I = \chi_I q_H - \chi_H q_I, \\ \overline{\omega}_I q_H - \overline{\omega}_H q_I = \chi_H p_I - \chi_I p_H. \end{cases} \tag{2}$$

2. Let us write $\quad p_I p_H + q_I q_H = E_{I,H},$

$$p_I q_H - p_H q_I = D_{I,H}.$$

* See my note: "*Sopra una estensione della teoria di Riemann sulle funzioni di variabile complessa.*" ("Atti della R. Acc. dei Lincei," Vol. III, fasc. 10.)

Among the E's and D's we shall have the relations

$$D_{IH}E_{LK} + D_{HK}E_{LI} + D_{KI}E_{LH} = 0, \tag{3}$$

$$\begin{vmatrix} E_{IH}E_{IL} \\ E_{KH}E_{KL} \end{vmatrix} = \begin{vmatrix} p_Ip_H + q_Iq_H & p_Ip_L + q_Iq_L \\ p_Kp_H + q_Kq_H & p_Kp_L + q_Kq_L \end{vmatrix} = \begin{vmatrix} p_Iq_I \\ p_Kq_K \end{vmatrix} \begin{vmatrix} p_Hq_H \\ p_Lq_L \end{vmatrix} = D_{IK}D_{HL}$$

whence

$$E_{IH}E_{KL} - E_{KH}E_{IL} = D_{IK}D_{HL}. \tag{4}$$

3. If we solve the equations (2) for $\bar{\omega}_I$ and χ_I, we shall have

$$\bar{\omega}_I = \frac{E_{IH}\chi_I - E_{II}\chi_H}{D_{HI}}, \qquad \chi_I = \frac{E_{IH}\bar{\omega}_I - E_{II}\bar{\omega}_H}{D_{IH}}$$

Since, however, the first member of these equations does not depend on H, we must have

$$\begin{aligned} \bar{\omega}_I &= \frac{E_{IH}\chi_I - E_{II}\chi_H}{D_{HI}} = \frac{E_{IK}\chi_I - E_{II}\chi_K}{D_{KI}} \\ &= \frac{(E_{IH}\chi_I - E_{II}\chi_H)E_{IK} - (E_{IK}\chi_I - E_{II}\chi_K)E_{IH}}{D_{HI}E_{IK} - D_{KI}E_{IH}} \\ &= \frac{E_{IH}\chi_K - E_{IK}\chi_H}{D_{HK}}. \end{aligned}$$

In a similar way we can operate on the expression for χ_I, and therefore whatever H and K may be we have the formulæ

$$\text{(E)} \qquad \bar{\omega}_I = \frac{E_{IH}\chi_K - E_{IK}\chi_H}{D_{HK}}, \quad \chi_I = \frac{E_{IH}\bar{\omega}_K - E_{IK}\bar{\omega}_H}{D_{KH}}.$$

4. From the preceding formulæ it follows that

$$D_{HK}\bar{\omega}_I = E_{IH}\chi_K - E_{IK}\chi_H,$$

$$D_{KI}\bar{\omega}_H = E_{HK}\chi_I - E_{HI}\chi_K,$$

$$D_{IH}\bar{\omega}_K = E_{KI}\chi_H - E_{KH}\chi_I,$$

hence, whatever I, H, K may be, we have the formula

$$D_{HK}\bar{\omega}_I + D_{KI}\bar{\omega}_H + D_{IH}\bar{\omega}_K = 0, \tag{F}$$

and similarly, $D_{HK}\chi_I + D_{KI}\chi_H + D_{IH}\chi_K = 0$.

5. Let us return to the equations (E); from them it follows that

$$\Theta_{IL} = \frac{1}{D_{IL}}\begin{vmatrix}\bar{\omega}_I & \chi_I\\ \bar{\omega}_L & \chi_L\end{vmatrix} = \frac{\chi_K E_{IH} - \chi_H E_{IK}}{D_{IL}D_{HK}}\chi_L - \frac{\chi_K E_{LH} - \chi_H E_{LK}}{D_{IL}D_{HK}}\chi_I \tag{5}$$

$$= \frac{E_{IH}\chi_K\chi_L - E_{IK}\chi_H\chi_L + E_{LK}\chi_H\chi_I - E_{LH}\chi_K\chi_I}{D_{IL}D_{HK}}.$$

If we interchange I with H and L with K the last member of this equation will not change. Hence we shall have

$$\frac{1}{D_{IL}}\begin{vmatrix}\bar{\omega}_I & \chi_I\\ \bar{\omega}_L & \chi_L\end{vmatrix} = \frac{1}{D_{HK}}\begin{vmatrix}\bar{\omega}_H & \chi_H\\ \bar{\omega}_K & \chi_K\end{vmatrix}. \tag{6}$$

In other words, the quantities Θ_{IL} are independent of I and L, and so we can denote them all by Θ.

If in (5) we put $I = H$, $L = K$, we shall have

$$\Theta = \frac{E_{II}\chi_L^2 - 2E_{IL}\chi_I\chi_L + E_{LL}\chi_I^2}{D_{IL}^2} \tag{7}$$

$$= \frac{(p_I\chi_L - p_L\chi_I)^2 + (q_I\chi_L - q_L\chi_I)^2}{D_{IL}^2}$$

formulæ which show that Θ is a *positive* quantity. If in (5) we interchange $\bar{\omega}$ and χ, and p and q, the Θ will not change, and we shall have for Θ the alternative expression

$$\Theta = \frac{E_{IH}\bar{\omega}_K\bar{\omega}_L - E_{IK}\bar{\omega}_H\bar{\omega}_L + E_{LK}\bar{\omega}_H\bar{\omega}_I - E_{LH}\omega_K\omega_I}{D_{IL}D_{HK}}. \tag{5'}$$

If we write $\phi = \phi_1 + i\phi_2$ and make use of our symbols $I, H \cdots$, we can write

$$\omega_I = \overline{\omega}_{i_1 \cdots i_{r+1}} = \frac{\partial \phi_1}{\partial (x_{i_1} x_{i_2} \cdots x_{i_{r+1}})} = \frac{\partial \phi_1}{\partial (x_I)},$$

$$\chi_1 = \chi_{i_1 \cdots i_{r+1}} = \frac{\partial \phi_2}{\partial (x_{i_1} x_{i_2} \cdots x_{i_{r+1}})} = \frac{\partial \phi_2}{\partial (x_I)}$$

where (x_I) is a substitute for $(x_{i_1} x_{i_2} \cdots x_{i_{r+1}})$, *i.e.*

$$(x_I) \equiv (x_{i_1} x_{i_2} \cdots x_{i_{r+1}}).$$

The expression for Θ can now be written

(G) $\Theta =$

$$\frac{E_{IH} \dfrac{\partial \psi}{\partial (x_I)} \dfrac{\partial \psi}{\partial (x_L)} - E_{IK} \dfrac{\partial \psi}{\partial (x_H)} \dfrac{\partial \psi}{\partial (x_K)} + E_{LK} \dfrac{\partial \psi}{\partial (x_H)} \dfrac{\partial \psi}{\partial (x_I)} - E_{LH} \dfrac{\partial \psi}{\partial (x_K)} \dfrac{\partial \psi}{\partial (x_I)}}{D_{IL} D_{HK}}$$

where in place of ψ we can put either ϕ_1 or ϕ_2.

6. We know that the quantities $\overline{\omega}$ and χ must satisfy the following equations (see section 5, article 1)

$$\sum_1^{r+2}{}_s (-1)^{s-1} \frac{\partial}{\partial x_s} \overline{\omega}_{t_1 \cdots t_{s-1} t_{s+1} \cdots t_{r+2}} = 0, \quad \sum_1^{r+2}{}_s (-1)^{s-1} \frac{\partial}{\partial x_s} \chi_{t_1 \cdots t_{s-1} t_{s+1} \cdots t_{r+2}},$$

and therefore, from (E), we have the following equations

(H)
$$\begin{cases} \displaystyle\sum_1^{r+2}{}_s (-1)^{s-1} \frac{\partial}{\partial x_s} \left\{ \frac{\chi_K E_{t_1 \cdots t_{s-1} t_{s+1} \cdots t_{r+2},\ H} - \chi_H E_{t_1 \cdots t_{s-1} t_{s+1} \cdots t_{r+2},\ K}}{D_{HK}} \right\} = 0 \\ \displaystyle\sum_1^{r+2}{}_s (-1)^{s-1} \frac{\partial}{\partial x_s} \left\{ \frac{\overline{\omega}_K E_{t_1 \cdots t_{s-1} t_{s+1} \cdots t_{r+2},\ H} - \overline{\omega}_H E_{t_1 \cdots t_{s-1} t_{s+1} \cdots t_{r+2},\ K}}{D_{HK}} \right\} = 0 \end{cases}$$

or, by reason of (H) and (F), ϕ_1 and ϕ_2 must satisfy the equations

$$(H') \quad \sum_{1}^{r+2}{}_{s}(-1)^{s-1}\frac{\partial}{\partial x_s}\left[\frac{E_{t_1\cdots t_{s-1}t_{s+1}\cdots t_{r+2},\ H}\dfrac{\partial\psi}{\partial(x_K)} - E_{t_1\cdots t_{s-1}t_{s+1}\cdots t_{r+2},\ K}\dfrac{\partial\psi}{\partial(x_H)}}{D_{HK}}\right] = 0$$

$$(F') \qquad D_{HK}\frac{\partial\psi}{\partial(x_I)} + D_{KI}\frac{\partial\psi}{\partial(x_H)} + D_{IH}\frac{\partial\psi}{\partial(x_K)} = 0.$$

7. Conversely it can be shown that if $\psi\,|\,[S_r]\,|$ is a real regular function and satisfies the preceding equations, it may be considered as the real part of a function $\psi + i\theta$ isogenous to f. In fact, by means of (H') we can write

$$\frac{E_{I,H}\dfrac{\partial\psi}{d(x_K)} - E_{I,K}\dfrac{\partial\psi}{\partial(x_H)}}{D_{HK}} = \frac{\partial\theta_{H,K}}{\partial(x_I)}$$

where $(x_I) = (x_{t_1}\cdots x_{t_{s-1}}x_{t_{s+1}}\cdots x_{t_{r+2}})$. But from (F') and (3) it follows that the first member of the preceding equations is independent of H and K, hence we can take the θ_{HK} as independent of their subscripts and write them all equal to θ, so that

$$\frac{E_{IH}\dfrac{\partial\psi}{\partial(x_K)} - E_{IK}\dfrac{\partial\psi}{\partial(x_H)}}{D_{HK}} = \frac{\partial\theta}{\partial(x_I)}.$$

And now if from these equations we follow the inverse procedure to that of articles 1, 2, 3, we find that the ratio

$$\frac{\dfrac{\partial(\psi + i\theta)}{\partial(x_I)}}{p_I + iq_I}$$

is independent of the indices (I), so that $\psi + i\theta$ is isogenous

to f. The equations (H') and (F') operate in our case in the same way as the equation $\Delta^2 = 0$ in the theory of Riemann.

7. *Conditions for isogeneity.*

1. If we take arbitrarily a regular function of hyperspaces S_r it will not always be possible to associate with it an isogenous function. In order for that it is necessary that certain conditions be satisfied. In fact if $F|[S_r]|$ is a regular function to which $\Phi|[S_r]|$ is isogenous, and we write

$$\frac{\partial F}{\partial(x_{i_1}\cdots x_{i_{r+1}})} = p_{i_1\cdots i_{r+1}}, \qquad \frac{\partial \Phi}{\partial(x_{i_1}\cdots x_{i_{r+1}})} = \bar{\omega}_{i_1\cdots i_{r+1}},$$

we must have

$$\frac{\bar{\omega}_{i_1\cdots i_{r+1}}}{p_{i_1\cdots i_{r+1}}} = \phi$$

where ϕ is independent of the indices $i_1 \cdots i_{r+1}$. Hence it follows that

$$\bar{\omega}_{i_1\cdots i_{r+1}} = \phi p_{i_1\cdots i_{r+1}}$$

so that

$$0 = \sum_1^{r+2}{}_s (-1)^s \frac{\partial \bar{\omega}_{i_1\cdots i_{s-1}i_{s+1}\cdots i_{r+2}}}{\partial x_{i_s}} = \sum_1^{r+2}{}_s (-1)^s \frac{\partial(\phi p_{i_1\cdots i_{s-1}i_{s+1}\cdots i_{r+2}})}{\partial x_{i_s}}$$

$$= \sum_1^{r+2}{}_s (-1)^s p_{i_1\cdots i_{s-1}i_{s+1}\cdots i_{r+1}} \frac{\partial \phi}{\partial x_{i_s}}.$$

From this we conclude that *it is necessary and sufficient in order that there may exist a function isogenous to* $F|[S_r]|$ *that the system of simultaneous linear differential equations*

$$(1) \qquad \sum_1^{r+2}{}_s (-1)^s p_{i_1\cdots i_{s-1}i_{s+1}\cdots i_{r+2}} \frac{\partial \phi}{\partial x_{i_s}} = 0$$

admit solutions.

It is for this reason that in § 9 we shall study systems of differential equations of this form. In the meantime let us observe that the equations (1) may in some cases be incompatible. Thus, if we have in four dimensions the regular function $F|[S_1]|$, the equations (1) become

$$-p_{23}\frac{\partial\phi}{\partial x_1}+p_{13}\frac{\partial\phi}{\partial x_2}-p_{12}\frac{\partial\phi}{\partial x_3}=0,$$

$$-p_{34}\frac{\partial\phi}{\partial x_2}+p_{24}\frac{\partial\phi}{\partial x_3}-p_{23}\frac{\partial\phi}{\partial x_4}=0,$$

$$-p_{41}\frac{\partial\phi}{\partial x_3}+p_{31}\frac{\partial\phi}{\partial x_4}-p_{34}\frac{\partial\phi}{\partial x_1}=0,$$

$$-p_{12}\frac{\partial\phi}{\partial x_4}+p_{42}\frac{\partial\phi}{\partial x_1}-p_{41}\frac{\partial\phi}{\partial x_2}=0,$$

and these equations will be incompatible unless

$$p_{12}p_{34}+p_{13}p_{42}+p_{14}p_{23}=0$$

2. We now proceed to prove the following theorem:

The necessary and sufficient condition in order that equations (1) *admit a common solution* ϕ *is that we can write*

$$(2)\qquad p_{i_1\cdots i_{r+2}}=\sum_1^{r+1}{}_s(-1)^s\frac{\partial\phi}{\partial x_{i_s}}\frac{\partial\psi}{\partial(x_{i_1}\cdots x_{i_{s-1}}x_{s+1}\cdots x_{i_{r+1}})}$$

where ψ *is a regular function of hyperspaces.*

Let us write $$\frac{\partial\psi}{\partial(x_{i_1}\cdots x_{i_r})}=q_{i_1\cdots i_r}.$$

It is easy to show that if the equations

$$(2')\qquad p_{i_1\cdots i_{r+1}}=\sum_1^{r+1}{}_s(-1)^s\frac{\partial\phi}{\partial x_{i_s}}q_{i_1\cdots i_{s-1}i_{s+1}\cdots i_{r+1}}$$

are satisfied, the equations (1) will also be satisfied. In fact, we shall have

$$\sum_1^{r+2}{}_t(-1)^t p_{i_1\cdots i_{t-1}i_{t+1}\cdots i_{r+2}}\frac{\partial\phi}{\partial x_{i_t}}$$
$$=\sum_1^{r+2}{}_t\sum_1^{r+2}{}_s{}^{(t)}(-1)^{s+t}\frac{\partial\phi}{\partial x_{i_s}}\frac{\partial\phi}{\partial x_{i_t}}q_{i_1\cdots i_{s-1}i_{s+1}\cdots i_{t-1}i_{t+1}\cdots i_{r+1}},$$

in which $\sum_1^{r+2}{}_s^{(t)}$ is extended over all the values of the index s from 1 to $r+2$, the value t excepted, and s' should be taken equal to s or to $s-1$ according as $s<t$ or $s>t$. Hence the left-hand member of the equation is zero, and the equations (1) are satisfied. From (2′) it also follows easily that

$$\sum_1^{r+2}{}_s (-1)^s \frac{\partial p_{i_1 \cdots i_{s-1} i_{s+1} \cdots i_{r+2}}}{\partial x_{i_s}} = 0.$$

Thus we have shown that our condition is sufficient. To show that it is also necessary, let us execute a change of variables, instead of $x_1, x_2 \cdots x_n$ taking $x_1' = \phi$, $x_2' = x_2 \cdots x_n' = x_n$. If we prime the letters which refer to the new variables, we shall have

1st) if $\quad i_1, i_2, \cdots i_r \neq 1$

$$q_{i_1 \cdots i_r} = q'_{i_1 \cdots i_r} + \sum_1^r{}_t (-1)^{t-1} q'_{i_t i_1 \cdots i_{t-1} i_{t+1} \cdots i_r} \frac{\partial \phi}{\partial x_{i_t}}$$

2d) if $\quad i_h = 1$

$$q_{i_1 \cdots i_r} = \sum_1^r{}_s (-1)^{s-1} q'_{1 i_1 \cdots i_{s-1} i_{s+1} \cdots i_r} \frac{\partial \phi}{\partial x_{i_s}} = (-1)^{h-1} q'_{1 i_1 \cdots i_{h-1} i_{h+1} \cdots i_r} \frac{\partial \phi}{\partial x_{i_h}}.$$

Supposing momentarily that $i_1 \cdots i_{r+1} \neq 1$ we shall have

$$p_{i_1 \cdots i_{r+1}} = \sum_1^{r+1}{}_s (-1)^s \frac{\partial \phi}{\partial x_{i_s}} q'_{i_1 \cdots i_{s-1} i_{s+1} \cdots i_{r+1}}$$

$$+ \sum_1^{r+1}{}_s (-1)^s \frac{\partial \phi}{\partial x_{i_s}} \sum_1^{r+1}{}_t{}^{(s)} (-1)^{t'} q'_{i_t i_1 \cdots i_{t-1} i_{t+1} \cdots i_{s-1} i_{s+1} \cdots i_{r+1}} \frac{\partial \phi}{\partial x_{i_t}}$$

where $t' = \begin{cases} t-1 \\ t \end{cases}$ according as $\begin{cases} t<s \\ t>s. \end{cases}$

Hence

$$(3) \qquad p_{i_1 \cdots i_{r+1}} = \sum_1^{r+1}{}_s (-1)^s q'_{i_1 \cdots i_{s-1} i_{s+1} \cdots i_{r+1}} \frac{\partial \phi}{\partial x_{i_s}}.$$

If we suppose instead that some one of the indices of p is equal to 1, say $i_1 = 1$, we shall have

$$(3') \qquad p_{1i_2\cdots i_{r+1}} = -\frac{\partial\phi}{\partial x_1}q'_{i_2\cdots i_{r+1}} - \frac{\partial\phi}{\partial x_1}\sum_2^{r+1}{}_t(-1)^t q'_{1i_2\cdots i_{t-1}i_{t+1}\cdots i_{r+1}}\frac{\partial\phi}{\partial x_{i_t}}$$
$$+\sum_2^{r+1}{}_s(-1)^s q'_{1i_2\cdots i_{s-1}i_{s+1}\cdots i_{r+1}}\frac{\partial\phi}{\partial x_1}\frac{\partial\phi}{\partial x_{i_s}} = -\frac{\partial\phi}{\partial x_1}q'_{i_2\cdots i_{r+1}}.$$

We shall show that (3) is a consequence of (3'). In fact, from (3') we have

$$q'_{i_2\cdots i_{r+1}} = -\frac{p_{1i_2\cdots i_{r+1}}}{\left(\dfrac{\partial\phi}{\partial x_1}\right)}$$

so that (3) becomes

$$p_{i_1\cdots i_{r+1}} = -\sum_1^{r+1}{}_s(-1)^s\frac{p_{1i_1\cdots i_{s-1}i_{s+1}\cdots i_{r+1}}}{\dfrac{\partial\phi}{\partial x_1}}\frac{\partial\phi}{\partial x_{i_s}}$$

and if we put $i_0 = 1$, this gives us

$$\sum_0^{r+1}{}_s(-1)^s p_{i_0 i_1\cdots i_{s-1}i_{s+1}\cdots i_r}\frac{\partial\phi}{\partial x_{i_s}} = 0,$$

an equation which is identically true.

We must now prove that the functions

$$q'_{i_2\cdots i_{r+1}} = -\frac{p_{1i_2\cdots i_{r+1}}}{\dfrac{\partial\phi}{\partial x_1}}$$

satisfy the conditions of integrability (see section 5, article 1), assuming therein that ϕ is constant.

We have in fact (see section 5, article 3)

$$p'_{1i_2\cdots i_{r+1}} = \frac{1}{\dfrac{d(\phi x_2\cdots x_n)}{d(x_1x_2\cdots x_n)}}\Sigma_h p_{h_1h_2\cdots h_{r+1}}\frac{d(x_{i_{r+2}}\cdots x_{i_n})}{d(x_{h_{r+2}}\cdots x_{h_n})}$$

where

$$(h_1h_2\cdots h_{r+1}h_{r+2}\cdots h_n) \equiv (1\,i_2\cdots i_{r+1}i_{r+2}\cdots i_n) \equiv (1,\,2,\cdots n)$$

so that

$$p'_{1i_2\cdots i_{r+1}} = \frac{p_{1i_2\cdots i_{r+1}}}{\left(\dfrac{\partial\phi}{\partial x_1}\right)}.$$

If $i_1, i_2 \cdots i_{r+1} \neq 1$, we have (see section 5, article 3)

$$p'_{i_1 \cdots i_{r+1}} = \frac{1}{\dfrac{d(\phi x_2 \cdots x_n)}{d(x_1 x_2 \cdots x_n)}} \Sigma_h p_{h_1 \cdots h_{r+1}} \frac{d(x_{i_{r+2}} \cdots x_{i_n})}{d(x_{h_{r+2}} \cdots x_{h_n})}$$

$$= \frac{1}{\dfrac{d\phi}{dx_1}} \sum_0^{r+1} (-1)^s p_{i_s i_1 \cdots i_{s-1} i_{s+1} \cdots i_{r+1}} \frac{\partial \phi}{\partial x_{i_s}} = 0.$$

And so if we apply the theorem of section 5, article 3, we shall have

$$0 = \sum_2^{r+1} (-1)^s \frac{\partial}{\partial x_{i_s}} \left[\frac{p_{1, i_2 \cdots i_{s-1} i_{s+1} \cdots i_{r+1}}}{\left(\dfrac{\partial \phi}{\partial x_1}\right)} \right]$$

$$= \sum_2^{r+1} (-1)^s \frac{\partial}{\partial x_{i_s}} q'_{i_2 \cdots i_{s-1} i_{s+1} \cdots i_{r+1}}.$$

The functions q' then satisfy the conditions of integrability, and it will be possible to determine a function ψ which satisfies equations (2). Thus it is shown that the given condition is necessary.

3. Given the F for which (1) is satisfied, the ψ which satisfies (2) is not determined. We shall see how all the ψ's which satisfy (2) may be found when one of them, ψ_1, is known. If ψ_1 and ψ satisfy (2), and we write

$$\psi - \psi_1 = \psi_2, \quad \frac{\partial \psi_2}{\partial (x_{i_1} \cdots i_r)} = q^{(2)}_{i_1 \cdots i}$$

we shall have

$$0 = \sum_1^{r+1} (-1)^s \frac{\partial \phi}{\partial x_{i_s}} q^{(2)}_{i_1 \cdots i_r}$$

and therefore

$$q^{(2)}_{i_1 \cdots i_r} = \sum_1^r (-1)^s \frac{\partial \phi}{\partial x_{i_s}} \frac{\partial \Theta}{\partial (x_{i_1} \cdots x_{i_{s-1}} x_{i_{s+1}} \cdots x_{i_r})},$$

in which $\Theta | [S_{r-1}] |$ is arbitrary.

THE GENERALIZATION OF ANALYTIC FUNCTIONS

Second Lecture

EXPRESSIONS FOR ISOGENOUS FUNCTIONS — AUXILIARY REMARKS ON SYSTEMS OF SIMULTANEOUS DIFFERENTIAL EQUATIONS — ON THE ELEMENTARY FUNCTIONS — COMPOSITION OF FUNCTIONS OF HYPERSPACES — NEW CONSIDERATIONS WITH REFERENCE TO THE RELATION OF ISOGENEITY — DIFFERENTIATION AND INTEGRATION — ISOGENEITY OF ORDER r.

8. *Expressions for isogenous functions*

1. If $F|[S_r]|$ and $\Phi|[S_r]|$ are isogenous, it follows from what has been shown in the preceding section that we can write

$$\frac{\partial F}{\partial(x_{i_1}\cdots x_{i_{r+1}})} = p_{i_1\cdots i_{r+1}} = \sum_1^{r+1}{}_s (-1)^s \frac{\partial f}{\partial x_{i_s}} \frac{\partial \psi}{\partial(x_{i_1}\cdots x_{i_{s-1}} x_{i_{s+1}}\cdots x_{i_{r+1}})}$$

$$\frac{\partial \Phi}{\partial(x_{i_1}\cdots x_{i_{r+1}})} = \bar{\omega}_{i_1\cdots i_{r+1}} = \sum_1^{r+1}{}_s (-1)^s \frac{\partial \phi}{\partial x_{i_s}} \frac{\partial \psi}{\partial(x_{i_1}\cdots x_{i_{s-1}} x_{i_{s+1}}\cdots x_{i_{r+1}})},$$

where $\psi|[S_{r-1}]|$ is regular and ϕ is a function of f; and we know that the ratio $\dfrac{\bar{\omega}_{i_1\cdots i_{r+1}}}{p_{i_1\cdots i_{r+1}}}$ (independent of the indices) is equal to $\dfrac{d\phi}{df}$.

2. Let us write

$$L_{i_1\cdots i_{r+1}} = f \frac{\partial \psi}{\partial(x_{i_1}\cdots x_{i_r})}.$$

It follows that

$$(1) \qquad p_{i_1\cdots i_{r+1}} = \sum_1^{r+1}{}_s (-1)^s \frac{\partial L_{i_1\cdots i_{s-1} i_{s+1}\cdots i_{r+1}}}{\partial x_{i_s}}.$$

If now S_{r+1} is a space of $r+1$ dimensions whose boundary is S_r, we shall have

$$F\,|\,[S_r]\,| = \int_{S_{r+1}} \Sigma_i p_{i_1 \cdots i_{r+1}} \alpha_{i_1 \cdots i_{r+1}} dS_{r+1},$$

where the $\alpha_{i_1 \cdots i_{r+1}}$ are the direction cosines of S_{r+1}. And if we substitute for the p's their values (1) and apply the extension of Stokes's theorem (see Section 4), we shall have

$$\mathrm{F}\,|\,[S_r]\,| = \int_{S_r} f \frac{d\psi}{dS_r} dS_r \tag{2}$$

and similarly,

$$\Phi\,|\,[S_r]\,| = \int_{S_r} \phi \frac{d\psi}{dS_r} dS_r. \tag{2'}$$

3. Conversely, if F and Φ are given by the preceding formulae, with $\phi = \phi(f)$, the F and Φ must be isogenous.

9. *Auxiliary remarks on systems of simultaneous differential equations*

1. Consider the system of differential equations

$$\mathrm{H}_{i_1 i_2 \cdots i_{r+2}} = \sum_1^{r+2} {}_s (-1)^s A_{i_1 \cdots i_{s-1} i_{s+1} \cdots i_{r+2}} \frac{\partial \phi}{\partial x_{i_s}} = 0 \tag{1}$$

whose coefficients satisfy the conditions

$$\sum_1^{r+2} {}_s (-1)^s A_{i_s h_1 \cdots h_{r+1}} A_{i_1 \cdots i_{s-1} i_{s+1} \cdots i_{r+2}} = 0 \tag{2}$$

and are such that they change sign with every transposition of the indices. With this convention, if we have an H with two of its indices equal, its value must be zero.

2. Among the A's, one at least must be different from zero. If $A_{i_1 i_2 \cdots i_{r+1}}$ is such a one, *all the equations* (1) *will follow from the equations* (*independent among themselves*).

$$(3) \qquad H_{i_1 \cdots i_{r+1} h_1} = 0, \quad H_{i_1 \cdots i_{r+1} h_2} = 0, \quad \cdots \quad H_{i_1 \cdots i_{r+1} h_{n-r-1}} = 0,$$

in which none of the $h_1, h_2 \cdots h_{n-r-1}$ *is equal to another, or to an* i.

Let us take, in fact, the system

$$(4) \qquad H_{i_1 \cdots i_{r+1} k_1} = 0, \quad H_{i_1 \cdots i_{r+1} k_2} = 0, \quad \cdots \quad H_{i_1 \cdots i_{r+1} k_{r+2}} = 0,$$

where the k_s are arbitrary. If a k_s is equal to one of the i_l, the corresponding equation will be an identity; otherwise, it will be one of the equations (3). The equations (4) can be written in the form

$$A_{i_1 \cdots i_{r+1}} \frac{\partial \phi}{\partial x_{k_s}} + \sum_1^{r+1}{}_l (-1)^l A_{k_s i_1 \cdots i_{l-1} i_{l+1} \cdots i_{r+1}} \frac{\partial \phi}{\partial x_{i_l}} = 0.$$

If we multiply each one by $(-1)^s A_{k_1 \cdots k_{s-1} k_{s+1} \cdots k_{r+2}}$ and add them together for all values of the subscript s from 1 to $r+2$, we shall have

$$A_{i_1 \cdots i_{r+1}} \sum_1^{r+2}{}_s (-1)^s A_{k_1 k_2 \cdots k_{s-1} k_{s+1} \cdots k_{r+2}} \frac{\partial \phi}{\partial x_{k_s}}$$
$$+ \sum_1^{r+1}{}_l (-1)^l \frac{\partial \phi}{\partial x_{i_l}} \sum_1^{r+2}{}_s A_{k_s i_1 \cdots i_{l-1} i_{l+1} \cdots i_{r+2}} A_{k_1 \cdots k_{s-1} k_{s+1} \cdots k_{r+2}} = 0,$$

whence $\sum_1^{r+2}{}_s (-1)^s A_{k_1 k_2 \cdots k_{s-1} k_{s+1} \cdots k_{r+2}} \frac{\partial \phi}{\partial x_{k_s}} = H_{k_1 \cdots k_{r+2}} = 0$

so that the theorem is proved.

3. Now let us form the alternating function of Poisson

$$(H_{i_1 i_2 \cdots i_{r+2}}, H_{h_1 h_2 \cdots h_{r+2}})$$

taking $i_1 = h_1$, $i_2 = h_2$, $\cdots$ $i_{r+1} = h_{r+1}$ and writing $h_{r+2} = i_{r+3}$,

we shall have

$$
\begin{aligned}
&(H_{h_1\cdots h_{r+2}},\ H_{t_1\cdots t_{r+2}})\\
&=\sum_1^{r+2}{}_t\sum_1^{r+2}{}_s(-1)^{s+t}\left(A_{h_1\cdots h_{s-1}h_{s+1}\cdots h_{r+2}}\frac{\partial A_{t_1\cdots t_{t-1}t_{t+1}\cdots t_{r+2}}}{\partial x_{h_s}}\frac{\partial\phi}{\partial x_{t_t}}\right)\\
&-\sum_1^{r+2}{}_s(-1)^{s+t}\left(A_{t_1\cdots t_{s-1}t_{s+1}\cdots t_{r+2}}\frac{\partial A_{h_1\cdots h_{t-1}h_{t+1}\cdots h_{r+2}}}{\partial x_{t_s}}\frac{\partial\phi}{\partial x_{ht}}\right)\\
&=\sum_1^{r+1}{}_t\sum_1^{r+1}{}_s(-1)^{s+t}\frac{\partial(A_{t_1\cdots t_{r+1}}A_{t_1\cdots t_{s-1}t_{s+1}\cdots t_{t-1}t_{t+1}\cdots t_{r+3}})}{\partial x_{t_s}}\frac{\partial\phi}{\partial x_{t_t}}\\
&-\sum_1^{r+2}{}_s(-1)^s\frac{\partial A_{h_1\cdots h_{s-1}h_{s+1}\cdots h_{r+2}}}{\partial x_{t_s}}\sum_1^{r+2}{}_t(-1)^tA_{t_1\cdots t_{t-1}t_{t+1}\cdots t_{r+3}}\frac{\partial\phi}{\partial x_t}\\
&+\sum_1^{r+2}{}_s(-1)^s\frac{\partial A_{t_1\cdots t_{s-1}t_{s+1}\cdots t_{r+2}}}{\partial x_{t_s}}\sum_1^{r+2}{}_t(-1)^tA_{h_1\cdots h_{t-1}h_{t+1}\cdots h_{r+2}}\frac{\partial\phi}{\partial x_{h_t}}\\
&+\sum_1^{r+2}{}_t(-1)^{t+r+2}\frac{\partial(A_{t_1\cdots t_{r+1}}A_{t_1\cdots t_{t-1}t_{t+1}\cdots t_{r+2}})}{\partial x_{t_{r+2}}}\frac{\partial\phi}{\partial x_{t_t}}\\
&-\sum_1^{r+2}{}_t(-1)^{t+r+2}\frac{\partial(A_{t_1\cdots t_{r+1}}A_{h_1\cdots h_{t-1}h_{t+1}\cdots h_{r+2}})}{\partial x_{t_{r+2}}}\frac{\partial\phi}{\partial x_{h_t}}\\
&+\sum_1^{r+2}{}_s(-1)^{s+r+2}\frac{\partial(A_{t_1\cdots t_{r+1}}A_{h_1\cdots h_{s-1}h_{s+1}\cdots h_{r+2}})}{\partial x_{h_s}}\frac{\partial\phi}{dx_{t_{r+2}}}\\
&-\sum_1^{r+2}{}_s(-1)^{s+r+2}\frac{\partial(A_{t_1\cdots t_{r+1}}A_{t_1\cdots t_{s-1}t_{s+1}\cdots t_{r+2}})}{\partial x_{t_s}}\frac{\partial\phi}{\partial x_{h_{r+2}}}.
\end{aligned}
$$

If we write

$$\sum_1^{r+2}{}_s(-1)^s\frac{\partial A_{h_1\cdots h_{s-1}h_{s+1}\cdots h_{r+2}}}{\partial x_{h_s}}=L_{h_1\cdots h_{r+2}},$$

we shall have

$$
\begin{aligned}
&(H_{h_1\cdots h_{r+2}}H_{t_1\cdots t_{r+2}})\\
&=A_{t_1\cdots t_{r+1}}\sum_1^{r+3}{}_s(-1)^sL_{t_1\cdots t_{s-1}t_{s+1}\cdots t_{r+3}}\frac{\partial\phi}{\partial x_{t_t}}\\
&+\sum_1^{r+3}{}_s(-1)^s\frac{\partial A_{t_1\cdots t_{r+1}}}{\partial x_t}H_{t_1\cdots t_{s-1}t_{s+1}\cdots t_{r+3}}+L_{t_1\cdots t_{r+2}}H_{t_1\cdots t_{r+1}t_{r+3}}\\
&\qquad\qquad -L_{t_1\cdots t_{r+1}t_{r+3}}H_{t_1\cdots t_{r+2}}.
\end{aligned}
$$

Hence to the system (1) we must add the equations

$$\sum_{1}^{r+1}{}_{s}(-1)^{s}L_{t_1\cdots t_{s-1}t_{s+1}\cdots t_{r+3}}\frac{\partial\phi}{\partial x_{t_t}}=0$$

so that *if the conditions*

$$L_{t_1\cdots t_{r+2}}=\sum_{1}^{r+2}{}_{s}(-1)^{s}\frac{\partial A_{t_1\cdots t_{s-1}t_{s+1}\cdots t_{r+2}}}{\partial x_{h_s}}=0$$

are satisfied for every combination of the indices $i_1\cdots i_{r+2}$, the *system* (1) *will be complete.*

4. From this it follows that the equations (1) of section 7 will form a complete system whenever, in addition to the conditions of integrability (see section 5, article 1), the functions p satisfy also the following conditions:

$$(5)\qquad \sum_{1}^{r+2}{}_{s}(-1)^{s}p_{t_1\cdots t_{s-1}t_{s+1}\cdots t_{r+2}}p_{t_sh_1\cdots h_r}=0.$$

Hence for *elementary* functions (see section 5, article 5) the system of equations (1) of section 7 is *complete.*

10. *The elementary functions*

1. Let us suppose that the function $F\,|\,[S_r]\,|$ is regular and elementary, so that the system (1) of section 7, or the equivalent system (3) of section 9, is complete. There will exist then $r+1$ independent integrals

$$\phi,\ \phi_1,\ \cdots\ \phi_r.$$

Hence the ratio

$$\theta=\frac{p_{t_1\cdots t_r}}{\left(\frac{d(\phi,\ \phi_1,\ \cdots\ \phi_r)}{d(x_{t_1}\cdots x_{t_{r+1}})}\right)}=\frac{\left(\frac{dF}{d(x_{t_1}\cdots x_{t_{r+1}})}\right)}{\left(\frac{d(\phi,\ \phi_1\cdots\phi_r)}{d(x_{t_1}\cdots x_{t_{r+1}})}\right)}$$

will be independent of the subscripts $i_1\cdots i_r$, and we shall have

$$p_{t_1\cdots t_{r+1}}=\theta\frac{d(\phi,\ \phi_1\cdots\phi_r)}{d(x_{t_1}\cdots x_{t_{r+1}})}.$$

But we must have $\sum_{1}^{r+2}{}_{s}(-1)^{s}\frac{\partial p_{t_1\cdots t_{s-1}t_{s+1}\cdots t_{r+2}}}{\partial x_{t_s}}=0,$

so that $\sum_{1}^{r+2}{}_{s}(-1)^{s}\frac{\partial\theta}{\partial x_{t_s}}\frac{d(\phi,\ \phi_1\cdots\phi_r)}{d(x_{t_1}\cdots x_{t_{s-1}}x_{t_{s+1}}\cdots x_{t_{r+2}})}=0,$

and consequently $\sum_{1}^{r+2}{}_{s}(-1)^{s}p_{t_1\cdots t_{s-1}t_{s+1}\cdots t_{r+2}}\frac{\partial\theta}{\partial x_{t_s}}=0.$

The quantity θ will therefore be a function of $\phi_0, \phi_1, \cdots \phi_r$, and if we write $\frac{\partial\phi_0}{\partial\phi}=\theta$, we shall have

$$p_{t_1\cdots t_{r+1}}=\frac{\partial\phi_0}{\partial\phi}\frac{d(\phi,\ \phi_1\cdots\phi_r)}{d(x_{t_1}\cdots x_{t_{r+1}})}=\frac{d(\phi_0,\ \phi_1\cdots\phi_r)}{d(x_{t_1}\cdots x_{t_{r+1}})}.$$

We have therefore the following theorem:

If F is an elementary function, it follows that

$$\frac{\partial F}{\partial(x_{t_1}\cdots x_{t_{r+1}})}=\frac{d(\phi_0,\ \phi_1\cdots\phi_r)}{d(x_{t_1}\cdots x_{t_{r+1}})}=p_{t_1\cdots t_{r+1}}$$

where $\phi_0, \phi_1, \cdots \phi_r$ are independent integrals of the complete system

(1) $$\sum_{1}^{r+2}{}_{s}(-1)^{s}p_{t_1\cdots t_{s-1}t_{s+1}\cdots t_{r+2}}\frac{\partial\phi}{\partial x_{t_s}}=0.$$

2. Conversely, *if we take $r+1$ functions $\phi_0, \phi_1, \cdots \phi_r$ and write*

$$\frac{d(\phi_0,\ \phi_1,\cdots\phi_r)}{d(x_{t_1}\cdots x_{t_{r+1}})}=p_{t_1\cdots t_{r+1}},$$

the quantities $p_{t_1\cdots t_{r+1}}$ will be the derivatives of an elementary function. In fact, they will satisfy the conditions of integrability, and also the conditions (5) of the preceding section (see section 5, article 5).

We shall say that the functions $\phi_0, \phi_1, \cdots \phi_r$ are *conjugate* to the function F, and that F is conjugate to them.

3. If Φ is isogenous to F, and we write

$$\frac{\partial \Phi}{\partial(x_{i_1} \cdots x_{i_{r+1}})} = \overline{\omega}_{i_1 \cdots i_{r+1}},$$

we must have

$$\frac{\overline{\omega}_{i_1 \cdots i_{r+1}}}{p_{i_1 \cdots i_{r+1}}} = \psi,$$

ψ being an integral of equation (1). Hence ψ must be a function of $\phi_0, \phi_1, \cdots \phi_r$. If we take $\psi = \frac{\partial \lambda}{\partial \phi}$, we shall have

$$\overline{\omega}_{i_1 \cdots i_{r+1}} = \frac{\partial \lambda}{\partial \phi} \frac{d(\phi, \phi_1, \cdots \phi_r)}{d(x_{i_1} \cdots x_{i_{r+1}})} = \frac{d(\lambda, \phi_1, \cdots \phi_r)}{d(x_{i_1} \cdots x_{i_{r+1}})},$$

from which we deduce the theorem:

All the functions isogenous to an elementary function are themselves elementary.

4. If we apply to the elementary functions the formula (2), section 8, relative to the possibility of defining isogenous functions, we have

$$F\,|\,[S_r]\,| = \int_{S_r} \phi \frac{d(\phi_0, \phi_1 \cdots \phi_r)}{d(\omega_1 \cdots \omega_r)} d\omega_1 \cdots d\omega_r, \tag{2}$$

where

$$x_1 = x_1(\omega_1, \cdots \omega_r), \quad x_2 = x_2(\omega_1, \cdots \omega_r), \cdots x_n = x_n(\omega_1, \cdots \omega_r),$$

the equations of the hyperspace S_r.

11. *The composition of functions of hyperspaces*

1. The results which we have obtained in the preceding section can be expressed in a different form by means of special symbols. That is what we shall do in this section, after having proved a fundamental theorem.

Let $F\,|\,[S_r]\,|$ and $\Phi\,|\,[S_{t-r}]\,|$ be two regular functions of

hyperspaces, and write

$$\frac{dF}{d(x_{h_1}\cdots x_{h_{r+1}})}=p_{h_1\cdots h_{r+1}},\quad \frac{\partial\phi}{\partial(x_{h_{r+2}}\cdots x_{h_{t+2}})}=q_{h_{r+2}\cdots h_{t+2}}.$$

$$(1)\qquad m_{i_1\cdots i_{t+2}}=\sum_h(-1)^{\binom{h_1\cdots h_{t+2}}{i_1\cdots i_{t+2}}}p_{h_1\cdots h_{r+1}}q_{h_{r+2}\cdots h_{t+2}},$$

in which $h_1\cdots h_{t+2}$ is a permutation of $i_1\cdots i_{t+2}$; the sum Σ_h is extended over all the combinations of the $t+2$ subscripts $i_1\cdots i_{t+2}$, $r+1$ at a time; and the symbol $(-1)^{\binom{h_1\cdots h_{t+2}}{i_1\cdots i_{t+2}}}$ represents $+1$ or -1, according as the substitution which appears in the exponent is even or odd.

2. We shall show that *there exists a regular function* $\Psi|[S_{t+1}]|$, *such that*

$$\frac{\partial\Psi}{\partial(x_{i_1}\cdots x_{i_{t+2}})}=m_{i_1\cdots i_{t+2}}.$$

In fact, the quantities m satisfy the conditions of integrability (section 5, article 1); that is,

$$\sum_1^{t+3}(-1)^s\frac{\partial m_{i_1\cdots i_{s-1}i_{s+1}\cdots i_{t+3}}}{\partial x_{i_s}}$$

$$=\sum_1^{t+3}(-1)^s\sum_h(-1)^{\binom{h_1\cdots h_{t+2}}{i_1\cdots i_{s-1}i_{s+1}\cdots i_{t+3}}}\frac{\partial}{\partial x_{i_s}}(p_{h_1\cdots h_{r+1}}q_{h_{r+2}\cdots h_{t+3}})=0.$$

3. To represent the fact that the relation (1) holds among the three functions F, Φ, Ψ we shall write

$$\Psi\equiv(F,\Phi).$$

We have immediately

$$(F,\Phi)\equiv(-1)^{(r+1)(t-r+1)}(\Phi,F).$$

If $\Theta\,|\,[S_{s-t}]\,|$ is a regular function, and we write

$$\frac{\partial\Theta}{\partial(x_{h_{t+3}}\cdots x_{h_{s+3}})} = n_{h_{t+3}\cdots h_{s+3}}$$

$$l_{i_1\cdots i_{s+3}} = \sum_h (-1)^{\binom{h_1\cdots h_{s+3}}{i_1\cdots i_{s+3}}} p_{h_1\cdots h_{r+1}} q_{h_{r+2}\cdots h_{t+2}} n_{h_{t+3}\cdots h_{s+3}}$$

$$= \sum_h (-1)^{\binom{h_1\cdots h_{s+3}}{i_1\cdots i_{s+3}}} m_{h_1\cdots h_{t+2}} n_{h_{t+3}\cdots h_{s+3}},$$

it follows that there exists a function $\Lambda\,|\,[S_{s+2}]\,|$ which is regular, and such that

$$\frac{\partial\Lambda}{\partial(x_{i_1}\cdots x_{i_{s+3}})} = l_{i_1\cdots i_{s+3}}.$$

We shall write $\Lambda = (F, \Phi, \Theta)$.

And in general if the functions $F^{(i)}\,|\,[S_{r_i}]\,|$ are regular, we shall understand by

(2) $$M \equiv (F^{(1)}, F^{(2)}, \cdots F^{(k)})$$

a regular function of hyperspaces S_R, $R = \sum_1^k r_i + k$, obtained as follows:

$$\Phi_2 \equiv (F^{(1)}, F^{(2)}),\ \Phi_3 \equiv (\Phi_2, F^{(3)}),\ \cdots M = (\Phi_{k-1} F^{(k)}).$$

We shall say that M is *composed* of the functions $F^{(1)}, F^{(2)}, \cdots F^{(k)}$ and we shall call the operation denoted by (2) the *composition* of the functions $F^{(1)}, F^{(2)}, \cdots F^{(k)}$. The operation of *composition* of the functions $F^{(i)}$ evidently possesses the *associative* property. Inversion of the elements of M can only produce changes in sign in the result.

The $F^{(i)}$ will be spoken of as the *divisors* of M. If M has no other divisors but itself, it will be spoken of as *prime*. If two functions have no *divisor* in common, they will be said to be *mutually prime*.

4. Without stopping to develop the theory of divisibility in the present sense, we can give directly a few of its proper-

ties and apply them to the results of the preceding sections. Thus, every regular function, which is not prime, can be decomposed into prime divisors, and this decomposition can be effected in more than one way. If a function divides one of the divisors of a function, it divides the function itself.

Two functions F and Φ will be isogenous when

$$F \equiv (\Psi, f), \quad \Phi = (\Psi, \phi),$$

where f, ϕ are point functions and f is a function of ϕ. If F and Φ are isogenous, so will be also the functions

$$(F, \Theta) \text{ and } (\Phi, \Theta).$$

No function is isogenous to a prime function; in order that a function may be found isogenous to a given function it is necessary and sufficient that the given function should admit a divisor which is a point function. That is, it is necessary for it to have the form $F \equiv (\Psi, f)$ with f a point function.

An elementary function is obtained by the composition of point functions, etc., etc.

12. *New considerations with reference to the relation of isogeneity*

1. So far we have been considering isogeneity between functions of hyperspaces of the same number of dimensions. We are now to generalize this relation so that it will apply to hyperspaces of different dimensions. Let us consider the two regular functions $\Phi|[S_r]|$, $\Psi|[S_t]|$, with $r > t$, and write

$$\frac{\partial \Phi}{\partial(x_{i_1} \cdots x_{i_{r+1}})} = a_{i_1 \cdots i_{r+1}} \quad , \quad \frac{\partial \Psi}{\partial(x_{i_1} \cdots x_{i_{t+1}})} = b_{i_1 \cdots i_{t+1}}.$$

We shall say that Φ and Ψ are isogenous when the following conditions are satisfied:

$$(1) \qquad \sum_1^{r+2}{}_s (-1)^s a_{i_1 \cdots i_{s-1} i_{s+1} \cdots i_{r+2}} b_{i_s h_1 \cdots h_t} = 0.$$

In the case where r is equal to t, these equations imply that the functions not only are isogenous in our first sense, but also that they are elementary. Conversely, if two elementary functions of hyperspaces of the same number of dimensions are isogenous in the sense of section 6, they are also in the present sense.

2. It is easy to show that *every function which admits* Φ *as divisor is isogenous to* Ψ. In fact, if we take

$$c_{i_1 \cdots i_{v+2}} = \sum (-1)^{\binom{h_1 \cdots h_{v+2}}{i_1 \cdots i_{v+2}}} a_{h_1 \cdots h_{r+1}} a'_{r+2, \cdots v+2},$$

we shall have $$\sum_{1}^{v+3} {}_{s} (-1)^s c_{i_1 \cdots i_{s-1} i_{s+1} \cdots i_{v+3}} b_{i_s h_1 \cdots h_t} = 0,$$

which proves the theorem.

3. We can now generalize a theorem given in section 7, article 2. We have:

The necessary and sufficient condition that $\phi \mid [S_r] \mid$ *shall be isogenous to the elementary function* $\Psi \mid [S_{r-t}] \mid$, *is that*

$$\Phi \mid [S_r] \mid = (\Psi, \Theta). \tag{2}$$

That the condition is sufficient can be shown without any difficulty. In order to show that it is also necessary, let us write

$$\frac{d\Phi}{d(x_{i_1} \cdots x_{i_{r+1}})} = a_{i_1 \cdots i_{r+1}}, \quad \frac{d\Psi}{d(x_{i_1} \cdots x_{i_{r-t+1}})} = b_{i_1 \cdots i_{r-t+1}},$$

$$\frac{d\Theta}{d(x_{i_1} \cdots x_{i_t})} = c_{i_1 \cdots i_t},$$

$$b_{i_1 \cdots i_{r-t+1}} = \frac{d(\phi_1, \phi_2 \cdots \phi_{r-t+1})}{d(x_{i_1} x_{i_2} \cdots x_{i_{r-t+1}})}.$$

We shall show that if (1) is true, (2) is also true; that is, that

$$a_{i_1 \cdots i_{r+1}} = \sum (-1)^{\binom{h_1 \cdots h_{r+1}}{i_1 \cdots i_{r+1}}} b_{h_1 \cdots h_{r-t+1}} c_{h_{r-t+2} \cdots h_{r+1}}. \tag{2'}$$

For this purpose let us make a change of variable, taking instead of $x_1, x_2, \cdots x_n$ the new variables $\phi_1, \phi_2, \cdots \phi_{t+1}$, $x_{t+2}, \cdots x_n$. If we indicate with a prime the symbols that belong with the new variables, we shall have

(i) If $h_{r-t+2}, h_{r-t+3}, \cdots h_r \neq \phi_1, \phi_2, \cdots \phi_{r-t+1}$, then

$$c_{h_{r-t+2}\cdots h_r} = c'_{h_{r-t+2}\cdots h_r} + \sum(-1)^{\binom{h_{p_1}\cdots h_{p_t}}{h_{r-t+2}\cdots h_{r+1}}} c'_{h_{p_1}\cdots h_{p_\nu} l_1 \cdots l_s} \frac{d(\phi_{l_1}\cdots\phi_{l_s})}{d\left(x_{h_{p_{\nu+1}}}\cdots x_{h_{p_{t-1}}}\right)},$$

in which $l_1, \cdots l_s$ are s of the numbers 1, 2, $\cdots r-t+1$, and $h_{p_1}\cdots h_{p_t}$ is a permutation of the numbers $h_{r-t+2}, \cdots h_{r+1}$.

(ii) If one of the numbers $h_{r-t+2}\cdots h_r$ is equal to one of the numbers 1, 2, $\cdots t+1$, then

$$c_{h_{r-t+2}\cdots h_r} = \sum(-1)^{\binom{h_{p_1}\cdots h_{p_{t-1}}}{h_{r-t+2}\cdots h_r}} c'_{h_{p_1}\cdots h_{p_\nu} l_1 \cdots l_s} \frac{d(\phi_{l_1}\cdots\phi_{l_s})}{d(x_{h_{p_{\nu+1}}}\cdots x_{h_{p_{t-1}}})}$$

Equation (2′) will then become

$$(2'')\quad a_{t_1\cdots t_{r+1}} = \sum(-1)^{\binom{h_1\cdots h_{r+1}}{t_1\cdots t_{r+1}}} b_{h_1\cdots h_{r-t+1}} c'_{h_{r-t+2}\cdots h_{r+1}} + \sum(-1)^{\binom{h_1\cdots h_{r+1}}{t_1\cdots t_{r+1}}} \frac{d(\phi_1\cdots\phi_{r-t+1})}{d(x_{h_1}\cdots x_{h_{r-t+1}})} \sum(-1)^{\binom{h_{p_1}\cdots h_{p_t}}{h_{r-t+2}\cdots h_{r+1}}} c'_{h_{p_1}\cdots h_{p_\nu} l_1\cdots l_s} \frac{d(\phi_{l_1}\cdots\phi_{l_s})}{d(x_{h_{p_{\nu+1}}}\cdots x_{h_{p_t}})},$$

in which the first sum is extended over all the possible combinations of the indices $h_{r-t+2}\cdots h_{r+1}$ which do not contain any of the numbers 1, 2, $\cdots r-t+1$. The second sum may be rewritten in the form

$$\sum(-1)^{\binom{h_{p_1}\cdots h_{p_{r+1}}}{t_1\cdots t_{r+1}}} c'_{h_{p_1}\cdots h_{p_\nu} l_1\cdots l_s} \sum(-1)^{h_{p_t}} \frac{d(\phi_1\cdots\phi_{r-t+1})}{d(x_{h_1}\cdots x_{h_{r-t+1}})} \frac{d(\phi_{l_1}\cdots\phi_{l_s})}{d(x_{h_{p_{\nu+1}}}\cdots x_{h_{p_{t-1}}})},$$

whence it vanishes. The equation (2″) reduces then to

$$(2''') \qquad a_{i_1 \cdots i_{r+1}} = \sum (-1)^{\binom{h_1 \cdots h_{r+1}}{i_1 \cdots i_{r+1}}} b_{h_1 \cdots h_{r-t+1}} c'_{h_{r-t+2} \cdots h_{r+1}}.$$

In particular we have

$$a_{1,2,\cdots t+1,\, t_{t+2} \cdots t_{r+1}} = b_{1,2,\cdots t+1} c'_{t_{t+2} \cdots t_{r+1}}$$

so that

$$(3) \qquad c'_{t_{t+2} \cdots t_{r+1}} = \frac{a_{1,2,\cdots t+1,\, t_{t+2} \cdots t_{r+1}}}{\left\{ \frac{d(\phi_1 \cdots \phi_{t+1})}{d(x_1 \cdots x_{t_{t+1}})} \right\}}.$$

Now by following a process analogous to that of section 7, article 2, it is easy to show that all the equations (2‴) are a consequence of these last equations (3). And so it is sufficient for us to show that the quantities c', obtained from (3), satisfy the conditions of integrability. We have in fact

$$a'_{1,\cdots t+1,\, t_{t+2},\cdots t_{r+1}} = \frac{a_{1,\cdots t+1,\, t_{t+2},\cdots t_{r+1}}}{\frac{d(\phi_1 \cdots \phi_{t+1})}{d(x_1 \cdots x_{t+1})}},$$

while a' will be zero if it has less than $t+1$ of its subscripts taken from the numbers 1, 2, $\cdots t+1$. If we apply then a process of reasoning analogous to that of section 7, article 2, we find that the conditions of integrability will be satisfied for the quantities c'.

13. *Differentiation and integration*

1. If two functions $F \mid [S_n] \mid$, $\Phi \mid [S_r] \mid$ are regular and isogenous, we know that the ratio

$$\phi = \frac{\left(\frac{d\Phi}{dS_{r+1}}\right)}{\left(\frac{dF}{dS_{r+1}}\right)}$$

will be independent of the hyperspace S_{r+1}, and will depend merely upon the point of the space at which the derivative is taken. The quantity ϕ will then be a point function for the total space of n dimensions. We shall denote it with the symbol $\frac{d\Phi}{dF}$ and call it the *derivative of* Φ *with respect to* F.

As a fundamental theorem it can be shown that *the derivative of* Φ *with respect to* F *is isogenous to both of the functions* Φ *and* F. The proof of this theorem comes immediately from formula (1) of section 7, with reference to the definition given in the preceding section.

2. Consider now a point function f isogenous to a regular function $F \mid [S_r] \mid$. By fixing the direction of the hyperspace S_{r+1} (see section 1, article 2) the quantity $\frac{dF}{dS_{r+1}}$ will be defined (see section 3, article 7), and hence the quantity

$$\int_{S_{r+1}} f \frac{dF}{dS_{r+1}} dS_{r+1}$$

will also be defined. This integral we shall represent by the symbol

$$\int_{S_{r+1}} f dF.$$

Changing the direction of the hyperspace will change the sign of the integral.

We shall suppose that the hyperspace S_{r+1} is closed and forms the boundary of a hyperspace S_{r+2} immersed in a portion of the total hyperspace S_n throughout which f and F have no singularities. It follows that

$$\int_{S_{r+1}} f dF = \int_{S_{r+1}} f \sum_i \frac{dF}{d(x_{i_1} \cdots x_{i_{r+1}})} \alpha_{i_1 \cdots i_{r+1}} dS_{r+1}$$

$$= \int_{S_{r+1}} f \sum_i p_{i_1 \cdots i_{r+1}} \alpha_{i_1 \cdots i_{r+1}} dS_{r+1},$$

where the $\alpha_{i_1 \cdots i_{r+1}}$ are the direction cosines of the hyperspace S_{r+1}. If we choose properly the direction of the hyperspace S_{r+2} and apply the generalization of Stokes's theorem (see section 4) we shall have

$$\int_{S_{r+1}} fdF = \int_{S_{r+2}} \sum \beta_{i_1 \cdots i_{r+2}} \sum (-1)^{s-1} \frac{\partial (f p_{i_1 \cdots i_{s-1} i_{s+1} \cdots i_{r+2}})}{\partial x_{i_s}} dS_{r+2}$$

$$= \int_{S_{r+2}} \sum \beta_{i_1 \cdots i_{r+2}} \left\{ \sum (-1)^s p_{i_1 \cdots i_{s-1} i_{s+1} \cdots i_{r+2}} \frac{\partial f}{\partial x_{i_s}} \right.$$

$$\left. + f \sum (-1)^{s-1} \frac{\partial p_{i_1 \cdots i_{s-1} i_{s+1} \cdots i_{r+2}}}{\partial x_{i_s}} \right\} dS_{r+2} = 0.$$

Hence we have the theorem expressed by the formula

$$\int_{S_{r+1}} fdF = 0. \tag{1}$$

If, instead of a single hyperspace S_{r+1} we have the hyperspaces $S_{r+1}^{(i)}$ $(i = 1, 2, \cdots n)$ which bound a space S_{r+2} within which there are no singularities for f or F, we shall have the formula:

$$\sum_1^n {}_i \int_{S_{r+1}^{(i)}} fdF = 0, \tag{1'}$$

in which the directions of the hyperspaces $S_{r+1}^{(i)}$ are all to be chosen with reference to the conventions adopted for the generalization of Stokes's theorem.

The theorem enunciated in the formulæ (1) *and* (1') *is the direct extension of Cauchy's theorem.*

3. Let us take away from the total hyperspace all those portions in which either f or F have singularities, and then introduce cuts in such a way that every closed hyperspace S_{r+1} may be taken as the complete boundary of a hyperspace S_{r+2}.

Take two hyperspaces S_r^0, S_r' such that a hyperspace S_{r+1} can be drawn to have them for its boundary, and choose the positive direction of S_r^0 and the negative direction of S_r' so as to correspond by the theorem of Stokes to one direction of the hyperspace S_{r+1}. With the direction of S_{r+1} fixed in this way, the integral

$$\int_{S_{r+1}} fdF \tag{2}$$

will be determined.

It is easy to show that the value of the integral (2) will not depend on the hyperspace S_{r+1}, but merely on S_r^0 and S_r. In fact if S_{r+1}' is another hyperspace which has these same two spaces for its boundary, the totality of S_{r+1} and S_{r+1}' will form a closed hyperspace, and from the hypotheses that we have made, we shall have

$$\int_{S_{r+1}+S_{r+1}'} fdF = 0,$$

from which the desired property follows.

Therefore the integral (2) can be indicated by the expression

$$\int_{S_r^0}^{S_r'} fdF. \tag{2'}$$

By changing the direction of S_{r+1} we change the sign of the integral; hence we may write

$$\int_{S_r^0}^{S_r'} fdF = -\int_{S_r'}^{S_r^0} fdF. \tag{3}$$

4. If we keep fixed the hyperspace S_r^0 and vary S_r', the integral (2') may be regarded as a function (regular) of S_r', and we can write

$$\int_{S_r^0}^{S_r'} fdF = \Phi\,|\,[S_r']\,|. \tag{4}$$

The function Φ *will be isogenous to* F *and we shall have*

$$\frac{d\Phi}{dF} = f, \tag{5}$$

that is to say, *the two operations of integration and differentiation are mutually inverse.*

14. *Isogeneity of order* r

1. A system of elementary functions will be said to have isogeneity of order r when all the functions of order greater than or equal to r, which are obtained from the system by means of composition (see section 11), vanish, while there is at least one function of order $r-1$ which does not vanish. All the elementary functions $\Phi|[S_i]|$ of the system must depend on certain functions $\phi_1, \phi_2, \cdots \phi_k, \cdots$ in such a way (see section 10) that

$$\frac{\partial\Phi}{\partial(x_{i_1}\cdots x_{i_{l+1}})} = \frac{d(\phi_{l_1}\cdots\phi_{l_{l+1}})}{d(x_{i_1}\cdots x_{i_{l+1}})},\quad \Phi \equiv (\phi_{l_1}, \phi_{l_2}, \cdots \phi_{l_{l+1}}).$$

2. We have immediately the following theorems:

The necessary and sufficient condition for isogeneity of order r *that is*

$$\frac{d(\phi_{l_1}\cdots\phi_{l_{r+1}})}{d(x_{i_1}\cdots x_{i_{r+1}})} = 0 \tag{1}$$

for every possible combination of the numbers $l_1, \cdots l_{r+1}, i_1, \cdots i_{r+1}$.

A function of order $r-1$ *is always isogenous to any other function of the system.*

In fact from (1) it follows that every function of order $r-1$ is isogenous to the functions of order zero of the system, that is, to the functions ϕ_s. We shall have

then

$$q_{i_s h_1 \cdots h_t} = \frac{\partial \Phi}{\partial (x_{i_s} x_{h_1} \cdots x_{h_t})} = \sum_{1}^{t+1}{}_h (-1)^{u-1} \frac{\partial \phi_{l_u}}{\partial x_{i_s}} \frac{d(\phi_{l_1} \cdots \phi_{l_{u-1}} \phi_{l_{u+1}} \phi_{l_{t+1}})}{d(x_{h_1} \cdots x_{h_t})}$$

$$= \sum_{1}^{t+1} (-1)^{u-1} \frac{\partial \phi_{l_u}}{\partial x_{i_s}} N_u.$$

And if we let $\psi \mid [S_{r-1}] \mid$ represent one of the functions of order $r - 1$ of the system, and write

$$\frac{\partial \psi}{\partial (x_{i_1} \cdots x_{i_r})} = p_{i_1 \cdots i_r},$$

we shall have

$$\sum_{1}^{r+1}{}_s (-1)^s p_{i_1 \cdots i_{s-1} i_{s+1} \cdots i_{r+1}} q_{i_s h_1 \cdots h_t}$$

$$= \sum_{1}^{t+1} (-1)^{u-1} N_u \sum_{1}^{r+1}{}_s (-1)^s p_{i_1 \cdots i_{s-1} i_{s+1} \cdots i_{r+1}} \frac{\partial \phi_{l_u}}{\partial x_{i_s}}.$$

Every function of order $r - 1$ admits as divisor another function of the system of lower order (see section 11, article 3).

3. Let us consider specially the functions of the system of order zero; that is, the functions $\phi_1, \phi_2, \cdots \phi_k, \cdots$. By means of the equations (1) we know that *there must be* r *of them, $\phi_1, \phi_2, \cdots \phi_r$, independent, of which all the others are functions*, and conversely, that *every function of $\phi_1, \phi_2, \cdots \phi_r$ will be an elementary function in the system, and will be of order zero.*

If we take two functions Φ and F of order $r - 1$, they will be isogenous, and we shall have the relation

$$\frac{d\Phi}{dF} = \phi(\phi_1, \phi_2, \cdots \phi_r). \tag{2}$$

Further, if we take an arbitrary function ϕ of order zero, that is a function of $\phi_1, \phi_2, \cdots \phi_r$, we shall have

$$\int_{S_r} \phi dF = 0, \tag{3}$$

where S_r is the complete boundary of a space S_{r+1} within which ϕ and F have no singularities. If we have

$$F \equiv (\phi_1, \phi_2, \cdots \phi_r),$$

then (3) can be written in the form

$$\int_{S_r} \phi \frac{d(\phi_1, \phi_2, \cdots \phi_r)}{d(\omega_1, \omega_2, \cdots \omega_r)} d\omega_1, d\omega_2, \cdots d\omega_r = 0,$$

$\omega_1, \cdots \omega_r$ being the parameters of the hyperspace S_r (see section 1, articles 1, 2). If we take

$$\frac{d\phi_t}{d\omega_s} d\omega_s = d_s\phi_t,$$

we shall have

$$\int_{S_r} \phi \begin{vmatrix} d_1\phi_1 & d_2\phi_1 \cdots d_r\phi_1 \\ d_1\phi_2 & d_2\phi_2 \cdots d_r\phi_2 \\ \cdot & \cdot \quad \cdot \quad \cdot \quad \cdot \\ d_1\phi_r & d_2\phi_r \cdots d_r\phi_r \end{vmatrix} = 0,$$

which is but a *generalization of Cauchy's theorem* (see the preceding section) put in a different form for the case of the elementary functions.

If S_r is not closed, but is bounded by two hyperspaces S^0_{r-1} and S_{r-1}, of which the first is fixed and the second variable, we shall have defined the expression

$$\Phi | [S_{r-1}] | = \int_{S^0_{r-1}}^{S_{r-1}} \phi \begin{vmatrix} d_1\phi_1 \cdots d_r\phi_1 \\ \cdot \quad \cdot \quad \cdot \quad \cdot \\ \cdot \quad \cdot \quad \cdot \quad \cdot \\ d_1\phi_r \cdots d_r\phi_r \end{vmatrix}.$$

BOOK OF THE OPENING

Third Lecture

ON THE THEORY OF WAVES AND GREEN'S METHOD*

Section 1

LET a homogeneous liquid be subjected to certain forces and let it occupy a domain S. Let this domain be limited by a frontier σ which is composed partly of a set ω' of rigid boundaries, and partly of a free surface ω, where the pressure is P.

Let us suppose that the state of equilibrium is stable. We shall study the small oscillations of the fluid when it is displaced from the state of equilibrium.

The hydrodynamical equations of Lagrange are

$$\left.\begin{aligned}
&\frac{d^2x}{dt^2}\cdot\frac{\partial x}{\partial x_0}+\frac{d^2y}{dt^2}\cdot\frac{\partial y}{\partial x_0}+\frac{d^2z}{dt^2}\cdot\frac{\partial z}{\partial x_0}=\frac{\partial}{\partial x_0}\left(V-\frac{P}{\rho}\right)\\
&\frac{d^2x}{dt^2}\cdot\frac{\partial x}{\partial y_0}+\frac{d^2y}{dt^2}\cdot\frac{\partial y}{\partial y_0}+\frac{d^2z}{dt^2}\cdot\frac{\partial z}{\partial y_0}=\frac{\partial}{\partial y_0}\left(V-\frac{P}{\rho}\right)\\
&\frac{d^2x}{dt^2}\cdot\frac{\partial x}{\partial z_0}+\frac{d^2y}{dt^2}\cdot\frac{\partial y}{\partial z_0}+\frac{d^2z}{dt^2}\cdot\frac{\partial z}{\partial z_0}=\frac{\partial}{\partial z_0}\left(V-\frac{P}{\rho}\right)
\end{aligned}\right\}\qquad(1)$$

where x, y, z, denote the coördinates of points of the fluid at time t, x_0, y_0, z_0 the initial coördinates, V the potential function, P the pressure, ρ the density.

2. Let x_0, y_0, z_0 be the coördinates which correspond to the state of stable equilibrium, ξ, η, ζ the components of displacement of each particle with respect to its position of equilibrium.

Then $\qquad x = x_0+\xi, \qquad y = y_0+\eta, \qquad z = z_0+\zeta.$

* Translated from the French by Professor Percy John Daniell, of the Rice Institute.

If we consider the displacements as infinitesimals of the first order and if we neglect terms of order higher than the first, the equations (1) become

$$\left.\begin{aligned}\frac{d^2\xi}{dt^2}&=\frac{\partial}{\partial x_0}\left(V-\frac{P}{\rho}\right)\\ \frac{d^2\eta}{dt^2}&=\frac{\partial}{\partial y_0}\left(V-\frac{P}{\rho}\right)\\ \frac{d^2\zeta}{dt^2}&=\frac{\partial}{\partial z_0}\left(V-\frac{P}{\rho}\right)\end{aligned}\right\}.$$

For simplification the indices o are suppressed and x, y, z denote the coördinates of each particle in the position of equilibrium.

Then

$$\left.\begin{aligned}\frac{d^2\xi}{dt^2}&=\frac{\partial}{\partial x}\left(V-\frac{P}{\rho}\right)\\ \frac{d^2\eta}{dt^2}&=\frac{\partial}{\partial y}\left(V-\frac{P}{\rho}\right)\\ \frac{d^2\zeta}{dt^2}&=\frac{\partial}{\partial z}\left(V-\frac{P}{\rho}\right)\end{aligned}\right\}. \tag{2}$$

The condition of incompressibility can be written as

$$\frac{\partial\xi}{\partial x}+\frac{\partial\eta}{\partial y}+\frac{\partial\zeta}{\partial z}=0. \tag{3}$$

On account of (2) we can put

$$\xi=\frac{\partial\Phi}{\partial x},\qquad \eta=\frac{\partial\Phi}{\partial y},\qquad \zeta=\frac{\partial\Phi}{\partial z},$$

Φ being the potential of displacement.

Then the equations (2) become

$$\frac{d^2\Phi}{dt^2}-V+\frac{P}{\rho}=c, \tag{4}$$

where c is constant with respect to x, y, z, but may vary with t.

The equation (3) becomes

$$\Delta^2\Phi = 0.$$

At points of the liquid where it touches the rigid boundary

$$\xi \cos nx + \eta \cos ny + \zeta \cos nz = 0,$$

if n denotes the normal to the boundary.

This condition becomes $$\frac{\partial\Phi}{\partial n} = 0.$$

3. Let us return to the equation (4). If we put

$$V - \frac{P}{\rho} + c = H,$$

the equation (4) becomes $$\frac{d^2\Phi}{dt^2} = H. \qquad (4')$$

The free surface of the fluid has been denoted by ω. Let us suppose that the potential function V and the pressure P, which correspond to each particle of fluid belonging to ω are functions of the coördinates of the point occupied by the particle independently of the form of the liquid. If this hypothesis is not correct, since the displacements are infinitesimal, we can neglect the variations produced by the changes in form of the fluid so that we can always proceed as if the hypothesis were correct.

In the state of equilibrium H is constant on ω. Therefore the equation of this surface will be

$$H = H_0 = \text{constant}.$$

Let us now calculate H when a point of the surface ω is displaced when ξ, η, ζ are the components of displacement.

If we neglect infinitesimals of a higher order than the first,

$$H = H_0 + \frac{\partial H}{\partial x}\xi + \frac{\partial H}{\partial y}\eta + \frac{\partial H}{\partial z}\zeta.$$

Then putting $\lambda^2 = \left(\frac{\partial H}{\partial x}\right)^2 + \left(\frac{\partial H}{\partial y}\right)^2 + \left(\frac{\partial H}{\partial z}\right)^2$,

$$\frac{\partial H}{\partial x} = \lambda \cos nx, \quad \frac{\partial H}{\partial y} = \lambda \cos ny, \quad \frac{\partial H}{\partial z} = \lambda \cos nz, \tag{5}$$

when n is the normal to the surface ω.

Then $$H = H_0 + \lambda (\xi \cos nx + \eta \cos ny + \zeta \cos nz)$$

$$= H_0 + \lambda \frac{\partial \Phi}{\partial n};$$

combining this with equation (4′)

$$\frac{\partial^2 \Phi}{\partial t^2} = H_0 + \lambda \frac{\partial \Phi}{\partial n}$$

or $$\frac{\partial^2 \Phi}{\partial t^2} = \lambda \frac{\partial \Phi}{\partial n},$$

since Φ is determinate except for a quantity which is constant with respect to the time.

Let us take the normal n as directed toward the interior of the fluid, and let us suppose that $V - \frac{P}{\rho}$ increases on moving ω and following the positive direction of n.

Then when n is positive, $\frac{\partial H}{\partial n} > 0$,

or by virtue of the equations (5)

$$\frac{\partial H}{\partial n} = \frac{\partial H}{\partial x} \cos nx + \frac{\partial H}{\partial y} \cos ny + \frac{\partial H}{\partial z} \cos nz = \lambda,$$

it follows that $\lambda > 0$.

The problem of waves can be presented in the following manner.

4. To determine a function Φ regular within the domain S which satisfies the equation

$$(A) \qquad \Delta^2 \Phi = 0$$

within S and which in the part ω' of the boundary satisfies the condition

$$(B) \qquad \frac{\partial \Phi}{\partial n} = 0$$

and in the part ω satisfies the condition

$$(C) \qquad \frac{\partial^2 \Phi}{\partial t^2} = \lambda \frac{\partial \Phi}{\partial n},$$

where λ is a positive quantity independent of the time, and n is the normal to the boundary directed toward the interior of the domain S.

Section 2

1. We can make a comparison between the problem we are about to consider and that of the vibrations of elastic media, and other problems of mathematical physics. The problem of the vibrations of elastic media is based upon the equation

$$\frac{\partial^2 u}{\partial t^2} = \alpha^2 \Delta^2 u. \qquad (6)$$

The problem of the propagation of heat in the case of varying temperature leads to the equation

$$\frac{\partial V}{\partial t} = a \Delta^2 V. \qquad (7)$$

The problems of potential and of stationary temperatures in isotropic bodies depend upon the equation of Laplace

$$\Delta^2 W = 0. \qquad (8)$$

These three equations are respectively of *hyperbolic*, *parabolic*, and *elliptic* types.

The question we have considered in section 1 belongs to the elliptic type on account of the equation (A) of section 1, which is the equation of Laplace; but it is the condition which must be satisfied on the surface ω of the boundary

which leads to the essential difference between this problem and the problems of potential and stationary temperatures. In fact, in the problems of potential the conditions at the boundary are reduced to that of giving the values of the unknown function or of its normal derivative; in those of stationary temperatures a linear relation between the unknown function and its normal derivative is known. But in the case of waves the condition at the boundary (equation (*C*) of section 1) introduces a new variable, the time, which makes the problem one of four variables. In respect to the number of variables the problem of waves is similar to the problems of vibrations and varying temperatures. It differs from them, however, because equations (6) and (7) have real characteristics. There are no real characteristics in the problem of the waves of liquids. We shall give a theorem in section 3 which will show the difference, from a physical standpoint, between waves in elastic media and waves in liquids.

2. There are two general methods in which the different problems we are investigating can be treated.

That of the separation of variables consists in separating the time from the space variables.

Let us put in the equation (6)

$$U = \sin mt \cdot u(x, y, z), \tag{9}$$

where m is a constant.

The equation becomes

$$m^2u + \alpha^2\Delta^2u = 0, \tag{10}$$

where the time has disappeared. If, for example, on the boundary $U = 0$, u must be taken $= 0$ there. We are led to find values of m for which the previous equation has solutions which are not identically zero (special solutions). The general solution is obtained by forming an infinite series of

solutions of the form (9) multiplied by arbitrary constants of such values that U and $\frac{\partial U}{\partial t}$ for $t = 0$ have the values of the given functions of x, y, z.

The question of determining the special solutions has been resolved by Poincaré; the theory of integral equations has been used and Mr. Hilbert, Mr. Schmidt, and others have founded the theory of series of special solutions.

Similarly an analogous process can be employed for equation (7) if we put $V = e^{mt}v(x, y, z)$; that is to say, by separating the time from the variables x, y, z.

Equation (7) reduces then to

$$mv + a\Delta^2 v = 0,$$

which is exactly analogous to equation (10).

3. The same method of the separation of the variables can be applied to the problem of waves in liquids.

If we put $\Phi = \sin mt\, \phi(x, y, z)$ equation (A) of section 1 becomes

$$\Delta^2\phi = 0,$$

equation (B) is

$$\frac{\partial\phi}{\partial u} = 0,$$

and equation (C) must be replaced by

$$m^2\phi + \lambda\frac{\partial\phi}{\partial n} = 0.$$

Here again the values of m corresponding to solutions ϕ which are not identically zero (special solutions) must be found.

By series of special solutions the general solution can be obtained. To calculate the values of m the method of Poincaré with those of integral equations can be used.

4. But we wish to set aside the process of the separation of variables and to pass on to the other general method. It

is the method which is connected with the ideas which Green used for the first time for the equation of Laplace and which, little by little, has been also used for other types of equations. By this point of view Kirchhoff arrived at his celebrated formula which expresses the principle of Huyghens. He applied Green's method to equation (6).

Betti has also applied an analogous method to equation (7).

We wish to show that a general formula can be found in the case of waves of fluids of a type which presents some analogies to these formulæ. I have had occasion to mention this formula without giving any development from it in my lectures at Stockholm. We shall now develop it and demonstrate in detail some applications of it.

Section 3

1. We shall begin by demonstrating in this paragraph some general theorems.

First Theorem. If Φ *is the function which satisfies the conditions* (A), (B), (C) *of section* 1, *it is determinate if the values* Φ_0, $\left(\frac{\partial\Phi}{\partial t}\right)_0$ *of* Φ *and* $\left(\frac{\partial\Phi}{\partial t}\right)$ *for* $t = 0$ *on the surface* ω *are known.*

Demonstration. Let Φ_1, Φ_2 be two functions which satisfy the conditions to which Φ is subjected.

Their difference $\Phi_3 = \Phi_1 - \Phi_2$ also satisfies the equations (A), (B), (C) and further we have

$$(\Phi_3)_0 = 0 \qquad \left(\frac{\partial\Phi_3}{\partial t}\right)_0 = 0$$

for $t = 0$ on the surface ω.

Let us now calculate

$$\Omega = \frac{1}{2}\frac{\partial}{\partial t}\int_\omega \frac{1}{\lambda}\left(\frac{\partial\Phi_3}{\partial t}\right)^2 d\omega.$$

On account of equation (*C*) we shall have

$$\Omega = \int_\omega \frac{1}{\lambda}\left(\frac{\partial \Phi_3}{\partial t}\right)\left(\frac{\partial^2 \Phi_3}{\partial t^2}\right) d\omega = \int_\omega \left(\frac{\partial \Phi_3}{\partial t}\right)\left(\frac{\partial \Phi_3}{\partial n}\right) d\omega.$$

But on ω' $\dfrac{\partial \Phi_3}{\partial n} = 0$ and therefore

$$\Omega = \int_\sigma \left(\frac{\partial \Phi_3}{\partial t}\right)\left(\frac{\partial \Phi_1}{\partial n}\right) \partial\sigma.$$

Applying a well-known transformation,

$$-\Omega = \int_S \left(\frac{\partial}{\partial x}\frac{\partial \Phi_3}{\partial t} \cdot \frac{\partial \Phi_3}{\partial x} + \frac{\partial}{\partial y}\frac{\partial \Phi_3}{\partial t} \cdot \frac{\partial \Phi_3}{\partial y} + \frac{\partial}{\partial z}\frac{\partial \Phi_3}{\partial t} \cdot \frac{\partial \Phi_3}{\partial z}\right) \partial S + \int_S \frac{\partial \Phi_3}{\partial t} \Delta^2 \Phi_3 dS.$$

The third term = 0; then

$$-\Omega = \frac{1}{2}\frac{\partial}{\partial t}\int_S \left\{\left(\frac{\partial \Phi_3}{\partial x}\right)^2 + \left(\frac{\partial \Phi_3}{\partial y}\right)^2 + \left(\frac{\partial \Phi_3}{\partial z}\right)^2\right\} dS$$

and it follows that

$$\frac{1}{2}\frac{\partial}{\partial t}\left[\int_\omega \frac{1}{\lambda}\left(\frac{\partial \Phi_3}{\partial t}\right)^2 d\omega + \int_S \left\{\left(\frac{\partial \Phi_3}{\partial x}\right)^2 + \left(\frac{\partial \Phi_3}{\partial y}\right)^2 + \left(\frac{\partial \Phi_3}{\partial z}\right)^2\right\} dS\right] = 0.$$

Integrating with respect to the time,

$$\int_\omega \frac{1}{\lambda}\left(\frac{\partial \Phi_3}{\partial t}\right)^2 d\omega + \int_S \left\{\left(\frac{\partial \Phi_3}{\partial x}\right)^2 + \left(\frac{\partial \Phi_3}{\partial y}\right)^2 + \left(\frac{\partial \Phi_3}{\partial z}\right)^2\right\} dS = c, \qquad (11)$$

where c is constant with respect to the time.

Then if $(\Phi_3)_0 = 0$ for $t = 0$ on ω, since $\dfrac{\partial \Phi_3}{\partial n} = 0$ on ω' $(\Phi_3)_0$ must be zero in the domain S. Consequently, the second integral in the formula (11) will be 0 for $t = 0$. In the same way, since $\left(\dfrac{\partial \Phi_3}{\partial t}\right)_0 = 0$, the first integral will be 0 for $t = 0$. It follows that $c = 0$, and the conclusion can be drawn that Φ_3 will be 0 for every value of t and therefore $\Phi_1 = \Phi_2$. Q. E. D.

2. *Second Theorem. If at a certain instant the molecules belonging to a part of the domain S are not displaced from the position of equilibrium, any molecule of the fluid is not displaced from the position of equilibrium.*

Demonstration. If ξ, η, ζ are 0 in any part of S, Φ will be constant in this part, and since it is an harmonic function regular in S, it will be everywhere constant. Consequently ξ, η, ζ will be 0 at all points of S. Q. E. D.

Third Theorem. If at a certain instant the molecules belonging to a part of the domain S are not displaced from the position of equilibrium and have no velocity, the fluid will remain always in the position of equilibrium.

Demonstration. If ξ, η, ζ and $\frac{d\xi}{dt}, \frac{d\eta}{dt}, \frac{d\zeta}{dt}$ are 0 in one part of the domain S at a certain instant, Φ and $\frac{d\Phi}{dt}$ will be constant in this part and therefore they will be constant in the whole domain S at the same instant. By virtue of the first theorem they will be constant in S at every instant and consequently the liquid will have no motion. Q. E. D.

3. These propositions show us the essential difference which exists between waves in liquids and waves in elastic media. In elastic media the motion is propagated with a certain velocity from one part to another; in liquids the motion reaches the whole mass contemporaneously, at least when the fluid does not remain in a constant state of equilibrium. In the case of liquids there is no propagation of motion and consequently one cannot speak of the velocity of propagation.

Section 4

1. Let Φ and Ψ be two functions which satisfy the conditions (A), (B), (C) of section 1.

By virtue of Green's theorem

$$\int_\sigma \left(\phi \frac{\partial \psi}{\partial n} - \psi \frac{\partial \phi}{\partial n}\right) d\sigma = 0$$

on account of (B) $\int_\omega \left(\phi \frac{\partial \psi}{\partial n} - \psi \frac{\partial \phi}{\partial n}\right) d\omega = 0.$

Using (C) this becomes

$$\int_\omega \left(\Phi \left(\frac{\partial^2 \psi}{\partial t^2} - \Psi \frac{\partial^2 \phi}{\partial t^2}\right) \frac{1}{\lambda} d\omega = 0. \qquad (12)$$

Let us now suppose that

$$\Psi = \frac{1}{r} + \chi,$$

where r denotes the distance between a point A (x_0, y_0, z_0) interior to the domain S and a point (x, y, z) and where χ is a regular function. Then the preceding formulæ are no longer valid for they presuppose that ψ is regular in the domain S. In this case formula (12) must be replaced by

$$4\pi \Phi_A + \int_\omega \left(\Phi \frac{\partial^2 \psi}{\partial t^2} - \psi \frac{\partial^2 \phi}{\partial t^2}\right) \frac{1}{\lambda} d\omega = 0, \qquad (12')$$

where Φ_A denotes the value of Φ at the point A.

Then $$4\pi \phi_A = -\frac{\partial}{\partial t} \int_\omega \left(\phi \frac{\partial \psi}{\partial t} - \psi \frac{\partial \phi}{\partial t}\right) \frac{1}{\lambda} \partial \omega.$$

Integrating between the limits 0 and t_1, we obtain

$$4\pi \int_0^{t_1} \phi_A dt = -\int_\omega \left\{ \phi_1 \left(\frac{\partial \psi}{\partial t}\right)_1 - \psi_1 \left(\frac{\partial \phi}{\partial t}\right)_1 \right\} \frac{1}{\lambda} \partial \omega$$

$$+ \int_\omega \left\{ \phi_0 \left(\frac{\partial \psi}{\partial t}\right)_0 - \psi_0 \left(\frac{\partial \psi}{\partial t}\right)_0 \right\} \frac{1}{\lambda} \partial \omega$$

where $\phi_1\ \psi_1\left(\frac{\partial\phi}{\partial t}\right)_1\left(\frac{\partial\psi}{\partial t}\right)_1$ denote the functions ϕ, ψ and the derivatives $\frac{\partial\phi}{\partial t}\frac{\partial\psi}{\partial t}$ for $t=t_1$, while $\phi_0\ \psi_0\left(\frac{\partial\phi}{\partial t}\right)_0\left(\frac{\partial\psi}{\partial t}\right)_0$ denote the same quantities for $t=t_0$. Let us now suppose that ψ_1 and $\left(\frac{d\psi}{dt}\right)_1$ are 0 on ω.

Then

$$(D)\qquad \Phi(x_0, y_0, z_0\, t_1)=\frac{1}{4\pi}\frac{d}{dt_1}\int_\omega\left\{\phi_0\left(\frac{\partial\psi}{\partial t}\right)_0-\psi_0\left(\frac{\partial\phi}{\partial t}\right)_0\right\}\frac{1}{\lambda}\,d\omega.$$

The above formula gives us a knowledge of Φ at every point in S and for every value of t when the values of ϕ_0 $\left(\frac{\partial\phi}{\partial t}\right)_0$ are known on ω. (Compare with the first theorem of section 3.)

It is necessary to calculate further the function Ψ and consequently χ. This function plays, in this case, *a part which can be compared with that played by Green's function.*

It must be remarked that ψ_0 and $\left(\frac{d\psi}{dt}\right)_0$ should depend on t_1 since ψ_1 and $\left(\frac{d\psi}{dt}\right)_1$ should be 0. The variable t_1 appears then in the second member of the equation (D) because it is contained in ψ_0 and $\left(\frac{d\psi}{dt}\right)_0$.

Section 5

In this paragraph we shall give some applications of the fundamental formula (D) of the preceding paragraph. Let us suppose that S is a sphere of radius R and that ω is the surface of the sphere in such a way that there are no rigid boundaries.

Let us put

$$\psi=a_0+\frac{(t_1-t)^2}{2!}a_2+\frac{(t_1-t)^4}{4!}a_4+\cdots,$$

$a_0, a_2, a_4 \cdots$ being coefficients independent of t_1 and t. We shall have

$$\psi_1 = a_0, \quad \left(\frac{d\psi}{dt}\right)_1 = 0.$$

But $$\psi = \frac{1}{r} + \chi.$$

$$\therefore a_0 = \frac{1}{r_A} + (\chi)_1,$$

and since a_0 should be 0 on ω and χ should be a regular and harmonic function if we use the method of images we obtain

$$(\chi)_1 = -\frac{R}{l}\frac{1}{r_{A'}},$$

where A' denotes the image point of A with respect to the sphere, $r_{A'}$ is the distance of the point A' from the point (x, y, z), l is the distance from the center of the sphere to the point A.

Then $$a_0 = \frac{1}{r_A} - \frac{R}{l}\frac{1}{r_{A'}}.$$

Let ρ be the radius vector, the pole being at the center of the sphere; then

$$\frac{\partial\psi}{\partial n} = -\frac{\partial\psi}{\partial\rho} = -\frac{\partial a_0}{\partial\rho} - \frac{(t_1-t)^2}{2!}\frac{\partial a_2}{\partial\rho} - \frac{(t_1-t)^4}{4!}\frac{\partial a_4}{\partial\rho}\cdots$$

$$\frac{\partial^2\psi}{\partial t^2} = a_2 + \frac{(t_1-t)^2}{2!}a_4 + \cdots.$$

Consequently on the surface ω, *i.e.* for $\rho = R$

$$-\lambda\frac{\partial a_0}{\partial\rho} = a_2, \quad -\lambda\frac{\partial a_2}{\partial\rho} = a_4, \quad -\lambda\frac{\partial a_4}{\partial\rho} = a_6, \cdots.$$

Since a_0 is known, the regular harmonic functions $a_2, a_4, a_6 \cdots$ must be determinate when their values on the boundary of the sphere are known.

Let us begin by transforming the expression for a_0. Let us denote by γ the angle between the lines joining the center of the sphere to the points A and (x, y, z).

Then $$a_0 = \frac{1}{(l^2+\rho^2-2\,l\rho\cos\gamma)^{\frac{1}{2}}} - \frac{R}{l}\frac{1}{\left(\frac{R^4}{l^2}+\rho^2-2\,\frac{R^2}{l}\,\rho\cos\gamma\right)^{\frac{1}{2}}}$$

or $$\frac{\rho}{R}\frac{\partial a_0}{\partial\rho} = \frac{\rho}{R}\frac{\partial}{\partial\rho}\left[\frac{1}{(l^2+\rho^2-2\,l\rho\cos\gamma)^{\frac{1}{2}}}\right] - \frac{\rho}{R}\frac{\partial}{\partial\rho}\left[\frac{R}{l}\cdot\frac{1}{\left(\frac{R^4}{l^2}+\rho^2-2\,\frac{R^2}{l}\,\rho\cos\gamma\right)^{\frac{1}{2}}}\right]$$

is a harmonic function which is equal to $\frac{\partial a_0}{\partial\rho}$ on the surface of the sphere; but it is not regular in the interior of the sphere. In fact, the first term of the second member becomes infinite for $\rho = l$, $\gamma = 0$. Then to calculate a_2 we cannot take the previous expression and multiply it by $-\lambda$ for a_2 must be regular in the interior of the sphere. But the following artifice may be used to calculate a_2.

Let us transform the first term of the second member by a transformation of reciprocal radii with respect to the sphere and let us multiply by $\frac{R}{\rho}$. The expression remains harmonic, possesses the same values on the boundary of the sphere, but becomes regular in the interior. To make the transformation of reciprocal radii it is sufficient to replace ρ by $\frac{R^2}{\rho}$. Thus the first term of the previous expression becomes

$$-R^2\frac{R^2-l\rho\cos\gamma}{(l^2\rho^2-R^4-2\,lR^2\rho\cos\gamma)^{\frac{3}{2}}}.$$

The second term equals

$$\frac{\rho l(l\rho-R^2\cos\gamma)}{(R^4+l^2\rho^2-2\,lR^2\rho\cos\gamma)^{\frac{3}{2}}}$$

It is found then that

$$a_2 = -\lambda \frac{l^2\rho^2 - R^4}{(R^4 + l^2\rho^2 - 2\,l\rho R^2 \cos\gamma)^{\frac{3}{2}}}.$$

In calculating a_4, $a_6 \cdots$ there are no more difficulties and

$$a_4 = -\lambda^2 \frac{\rho}{R}\frac{\partial}{\partial\rho}\left[\frac{R^4 - l^2\rho^2}{R^4 + l^2\rho^2 - 2\,l\rho R^2\cos\gamma)^{\frac{3}{2}}}\right]$$

$$= -\frac{\lambda^2}{R}\frac{\partial}{\partial \log\rho}\left[\frac{R^4 - l^2\rho^2}{(R^4 + l^2\rho^2 - 2\,l\rho R^2\cos\gamma)^{\frac{3}{2}}}\right].$$

In general,

$$a_{2n} = (-1)^{n-1}\frac{\lambda^n}{R^{n-1}}\frac{\partial^{n-1}}{\partial(\log\rho)^{n-1}}\left[\frac{R^4 - l^2\rho^2}{R^4 + l^2\rho^2 - 2\,l\rho R^2\cos\gamma)^{\frac{3}{2}}}\right].$$

Consequently,

$$\Psi = a_0 + \sum_1^\infty (-1)^{n-1}\frac{\lambda^n}{R^{n-1}}\frac{\partial^{n-1}}{\partial(\log\rho)^{n-1}}$$

$$\left[\frac{R^4 - l^2\rho^2}{R^4 + l^2\rho^2 - 2\,l\rho R^2\cos\gamma)^{\frac{3}{2}}}\right]\frac{(t_1 - t)^{2n}}{2\,n\,!}.$$

$$\frac{\partial\Psi}{\partial t} = -\sum_1^\infty (-1)^{n-1}\frac{\lambda^n}{R^{n-1}}\frac{\partial^{n-1}}{\partial(\log\rho)^{n-1}}$$

$$\left[\frac{R^4 - l^2\rho^2}{(R^4 + l^2\rho^2 - 2\,l\rho R^2\cos\gamma)^{\frac{3}{2}}}\right]\frac{(t_1 - t)^{2n-1}}{(2\,n-1)\,!}.$$

In order to calculate the formula (D) of section 4 it is necessary to evaluate ψ_0 and $\left(\frac{d\psi}{dt}\right)_0$, that is to say, to put $t = 0$ in the previous series. Further it is the values at the surface of the sphere which have to be found. Finally, this expression must be derived with respect to t_1.

Let us then adopt polar coördinates and put

$$x = \rho\sin\theta\cos\phi, \qquad y = \rho\sin\theta\sin\phi, \qquad z = \rho\cos\theta,$$
$$x_0 = l\sin\theta_0\cos\phi_0, \qquad y_0 = l\sin\theta_0\sin\phi_0, \qquad z = l\cos\theta_0.$$

Then $\cos\gamma = \cos\phi\cos\phi_0 + \sin\phi\sin\phi_0\cos(\theta - \theta_0)$.

Let us write

$$\Theta(l, \theta_0, \phi_0, \theta, \phi, t) = \sum_1^\infty (-1)^{n-1} \frac{\lambda^{n-1}}{R^{n-1}} \frac{\partial^{n-1}}{\partial(\log l)^{n-1}} \left[\frac{R^2 - l^2}{R^2 + l^2 - 2Rl\cos\gamma)^{\frac{3}{2}}}\right] \frac{t^{2n-1}}{(2n-1)!}.$$

Formula (D) can be written

$$(D_a)\Phi(l, \theta_0, \phi_0, t) = \frac{R}{\gamma\pi}\int_\omega \phi_0'(\theta, \phi)\Theta(l, \theta_0, \phi_0, \theta, \phi, t)\sin\theta d\theta d\phi + \frac{R}{\gamma\pi}\frac{d}{dt}\int_\omega \phi_0(\theta, \phi)\Theta(l, \theta_0, \phi_0, \theta, \phi, t)\sin\theta d\theta d\phi,$$

where for simplification we have written

$$\Phi_0(\theta, \phi) = \phi_0(R, \theta, \phi, t), t = 0$$

$$\Phi_0'(\theta, \phi) = \left\{\frac{d}{dt}\phi_0(R, \theta, \phi, t)\right\}, t = 0.$$

The formula we have been seeking to find is the general formula in the case of the sphere.

If, instead of a sphere, the liquid occupies a hemisphere and the diametral plane constitutes the rigid boundary so that the curved surface is free, the method of images will provide the solution in a similar manner. The same holds in the case where the liquid occupies a section of a sphere between two rigid diametral planes the angle between which equals $\frac{\pi}{n}$, where n is an integer.

VITO VOLTERRA.

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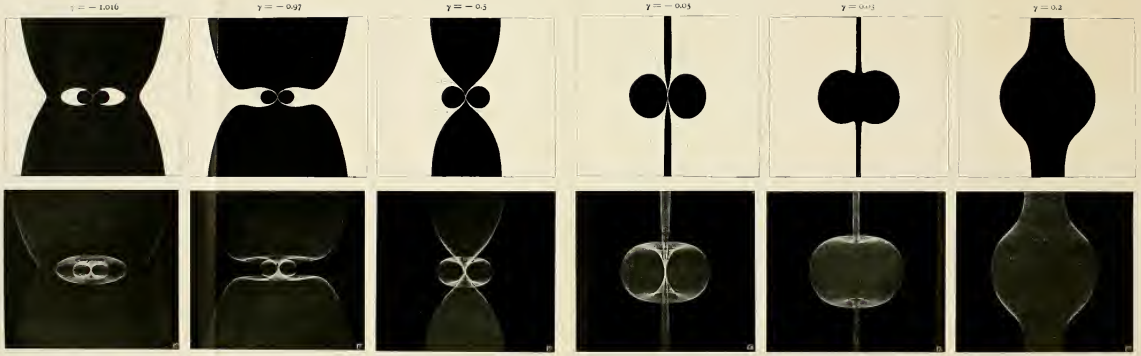


PLATE I

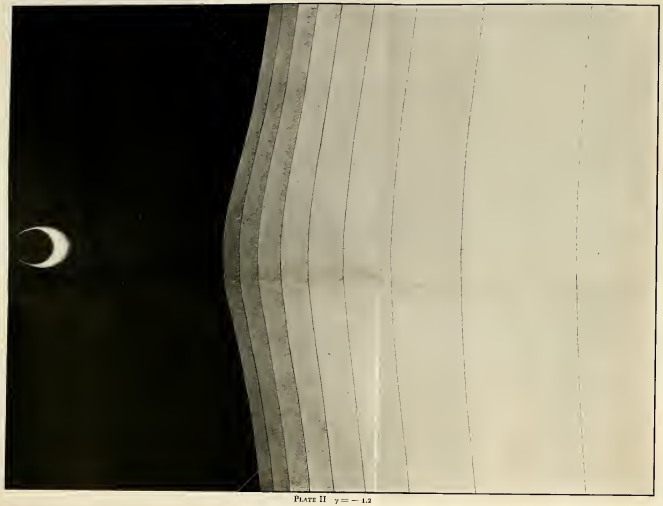


PLATE II $\gamma = -1.2$

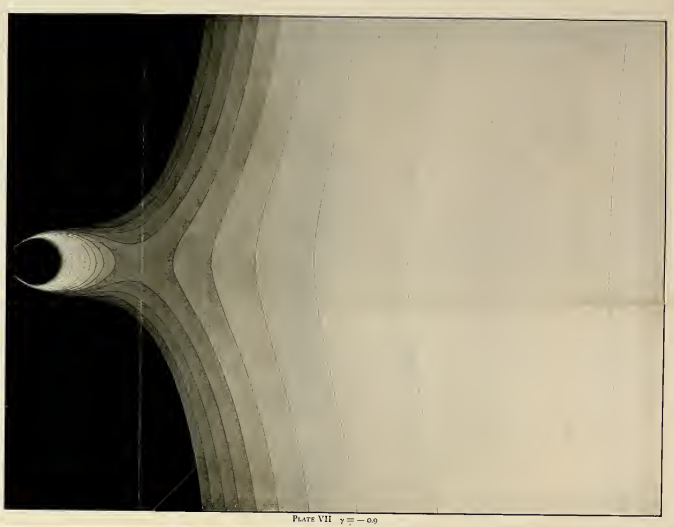


PLATE VII $\gamma = -0.9$

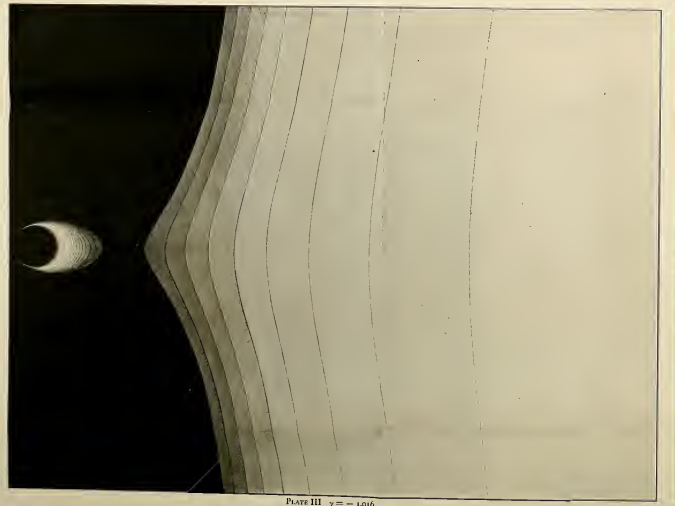


PLATE III $\gamma = -1.016$

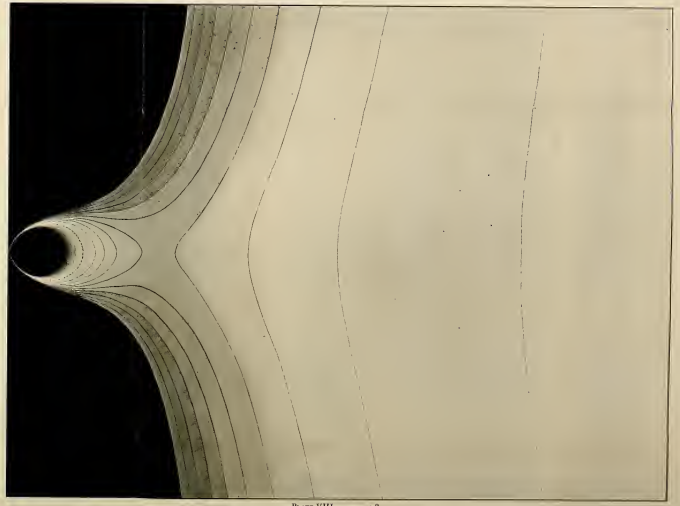


PLATE VIII $\gamma = -0.8$

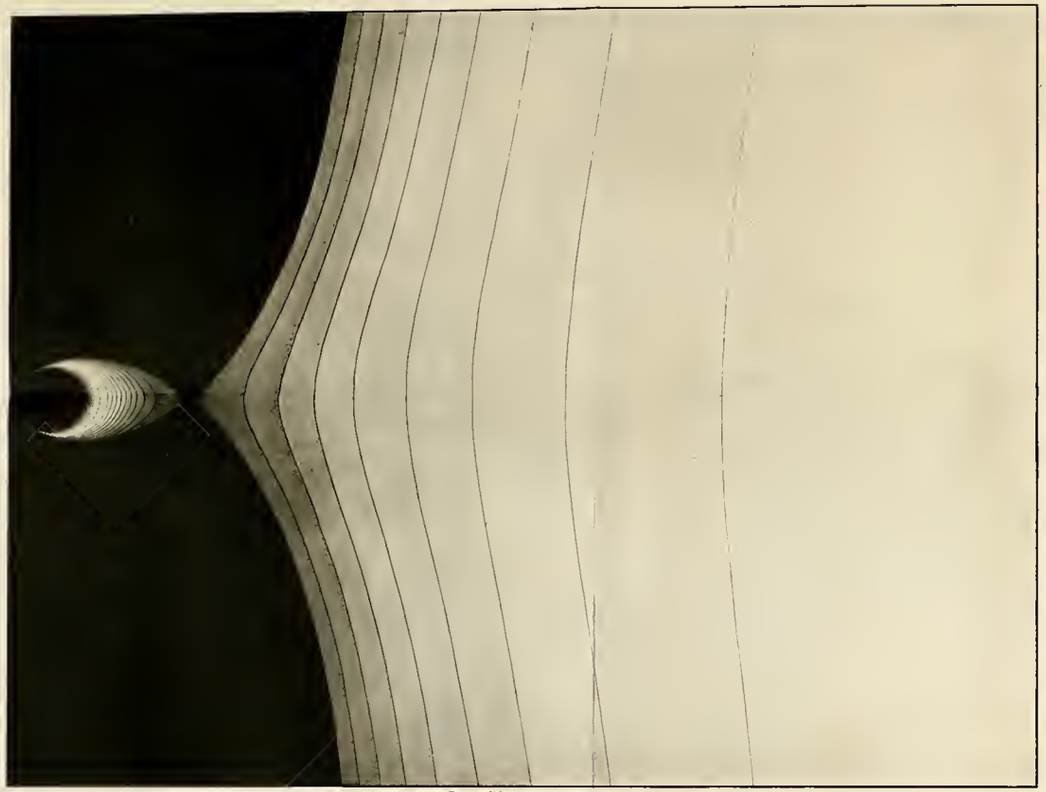


PLATE IV $\gamma = -1.001$

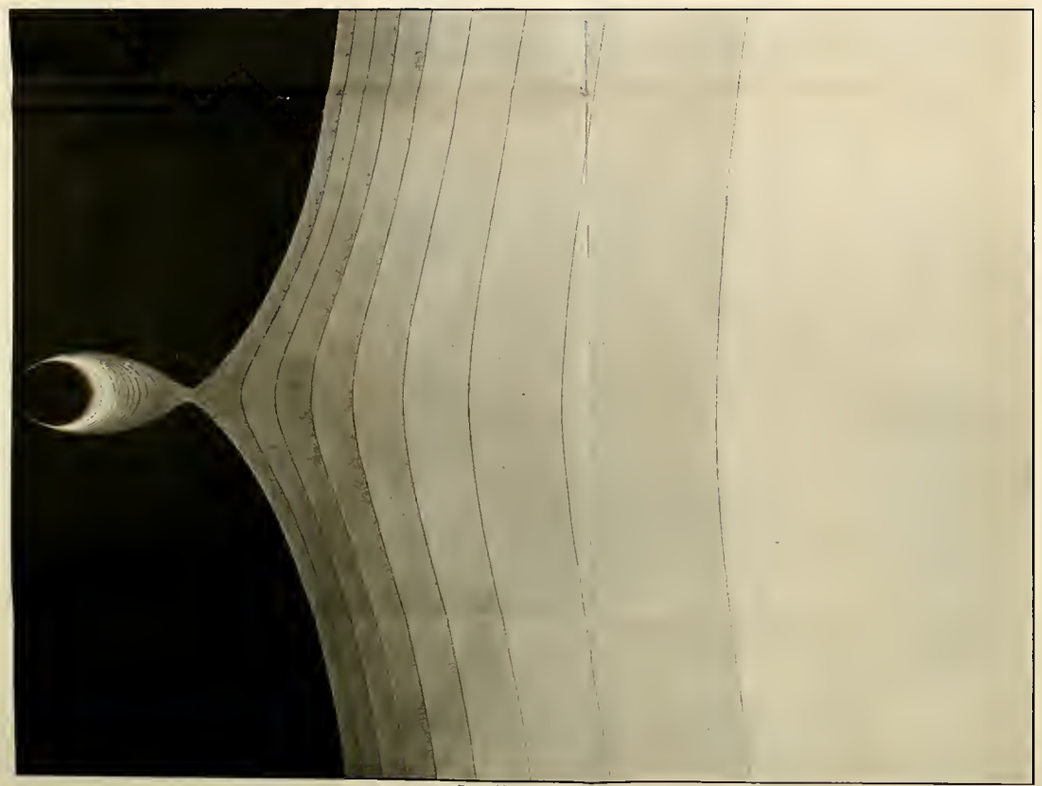


PLATE V $\gamma = -0.999$

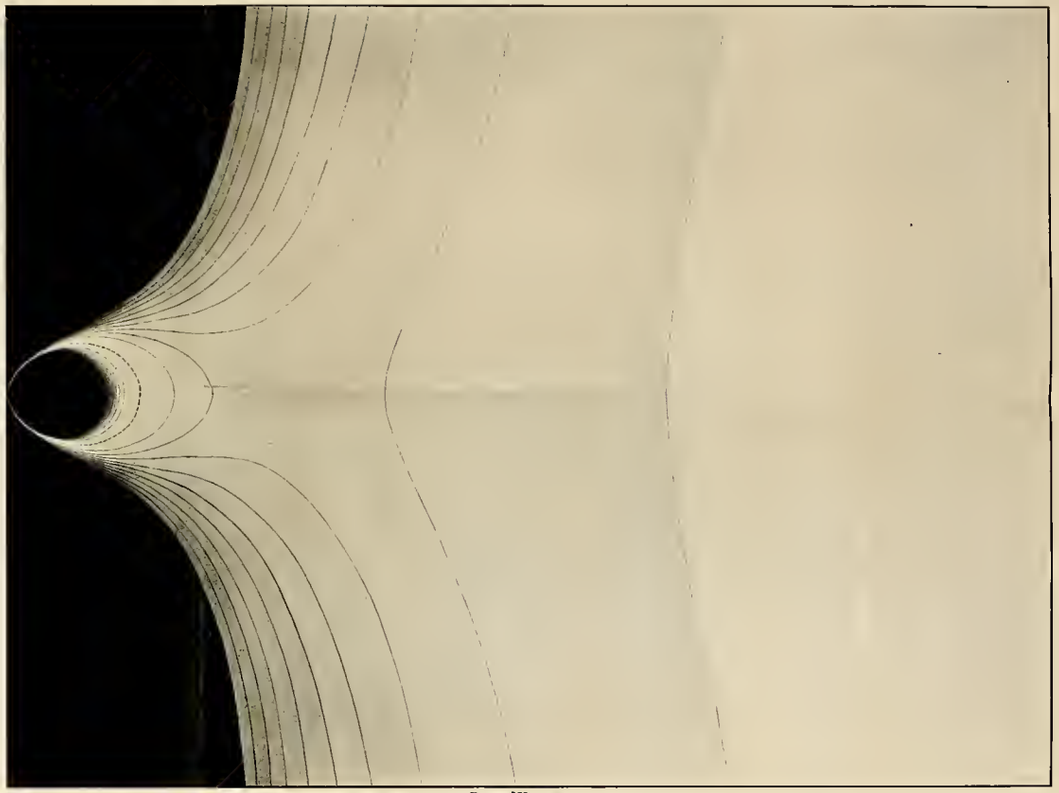


PLATE IX $\gamma = -0.7$

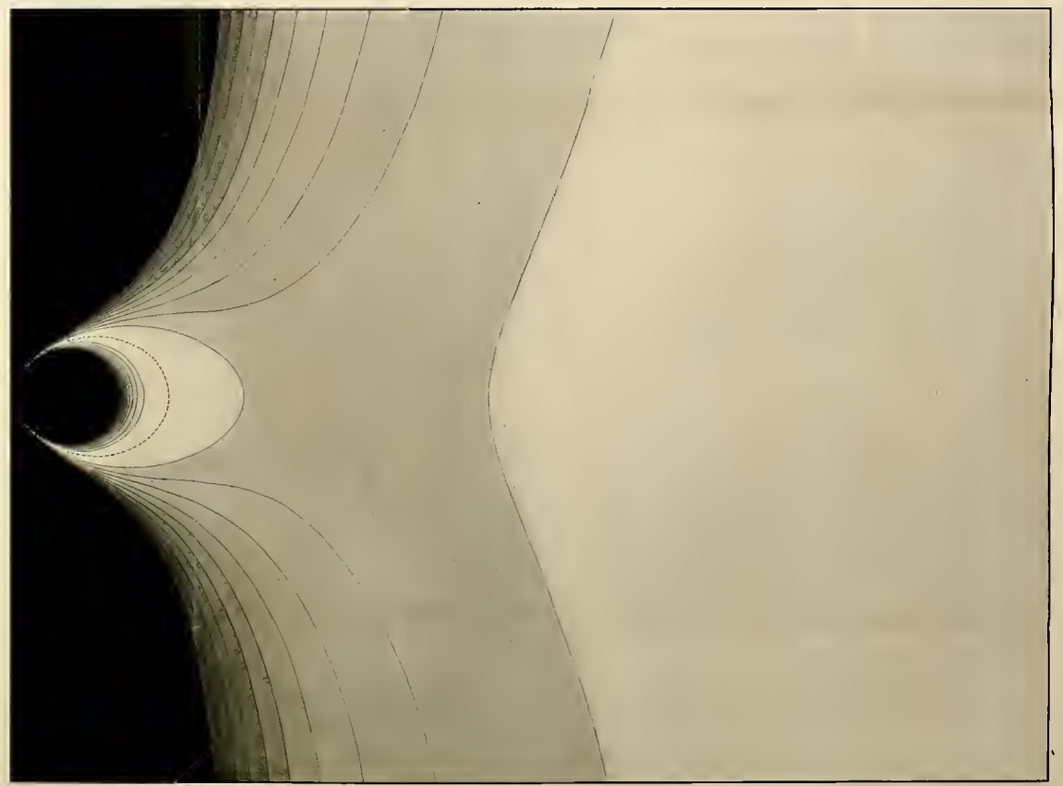


PLATE X $\gamma = -0.6$

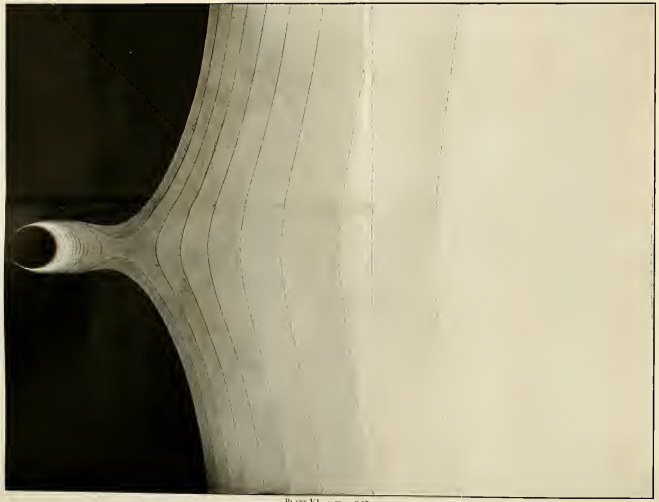


PLATE VI $\gamma = -0.97$

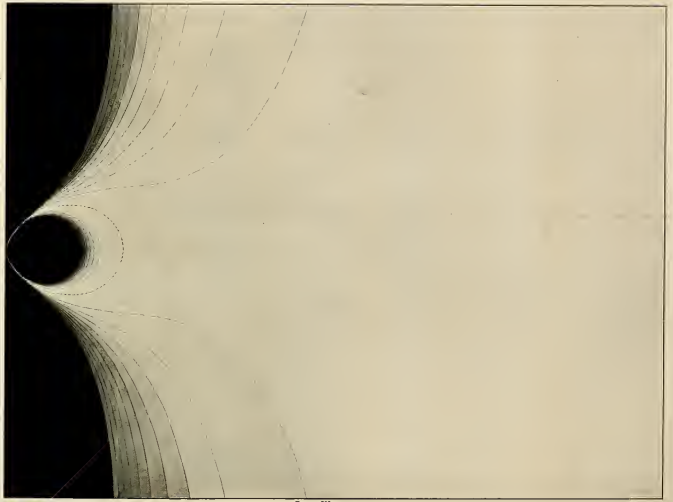


PLATE XI $\gamma = -0.5$

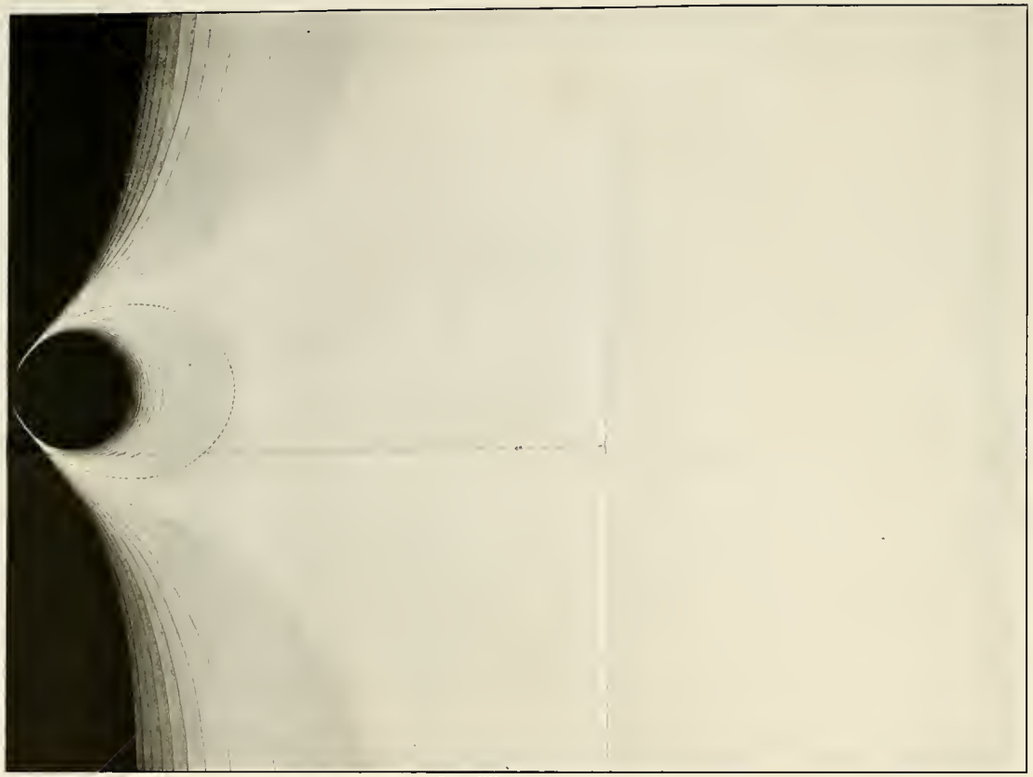


PLATE XII $\gamma = -0.4$

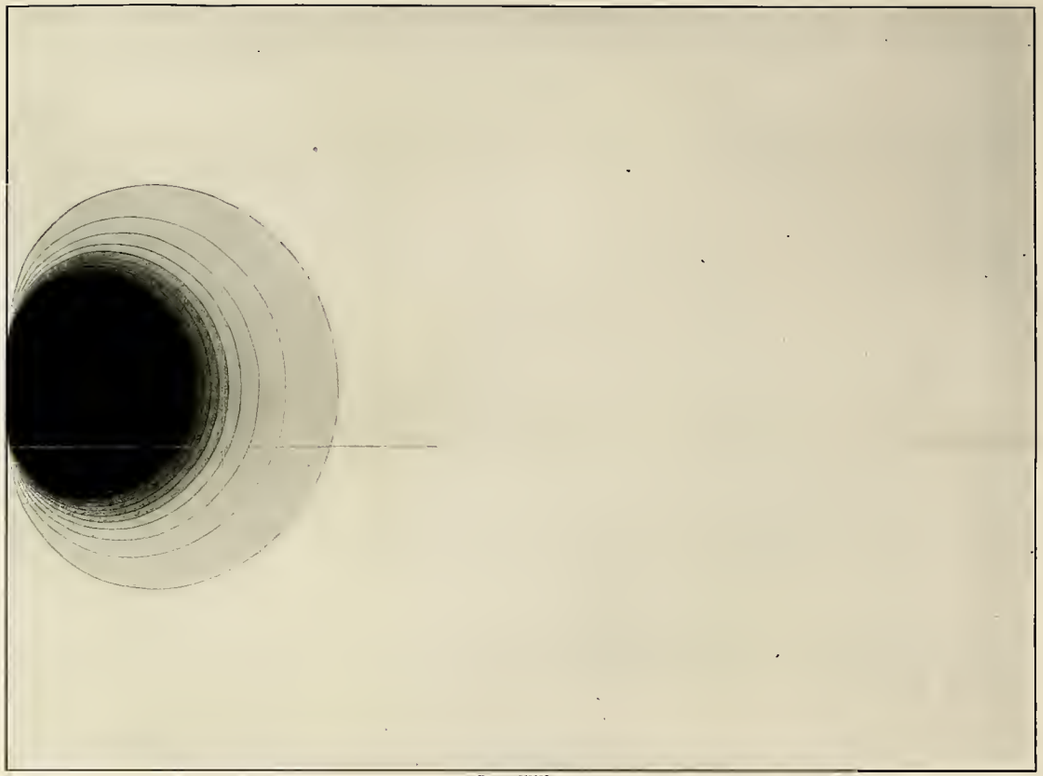


PLATE XVII $\gamma = 0$

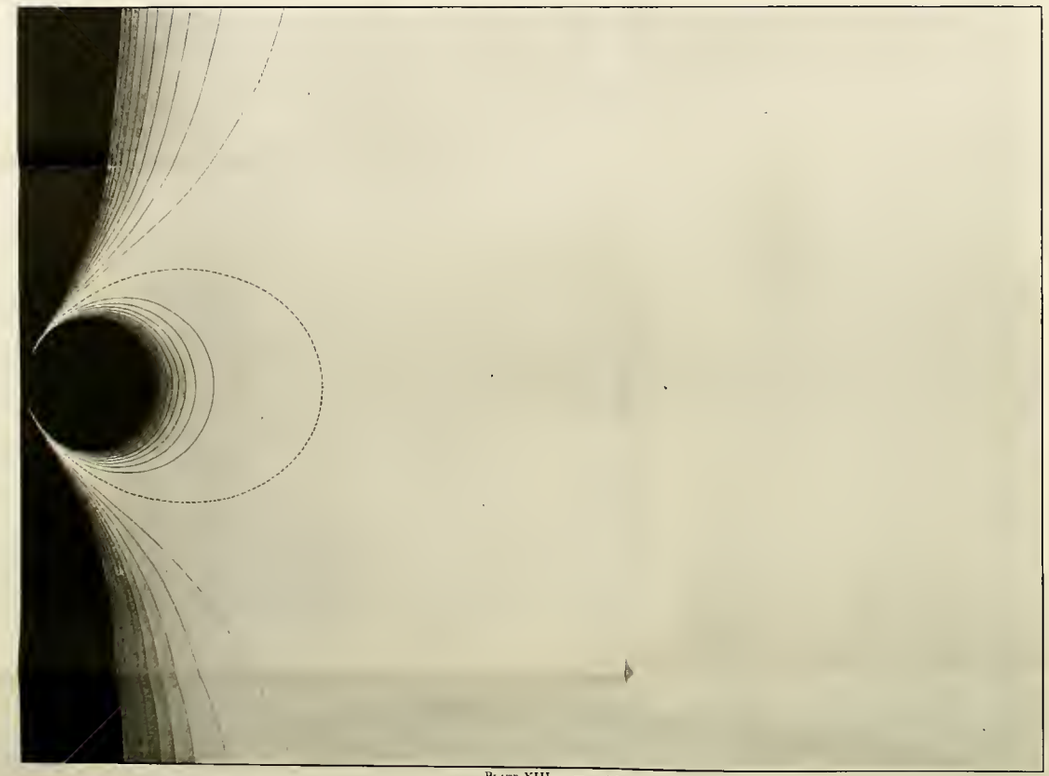


PLATE XIII $\gamma = -0.3$

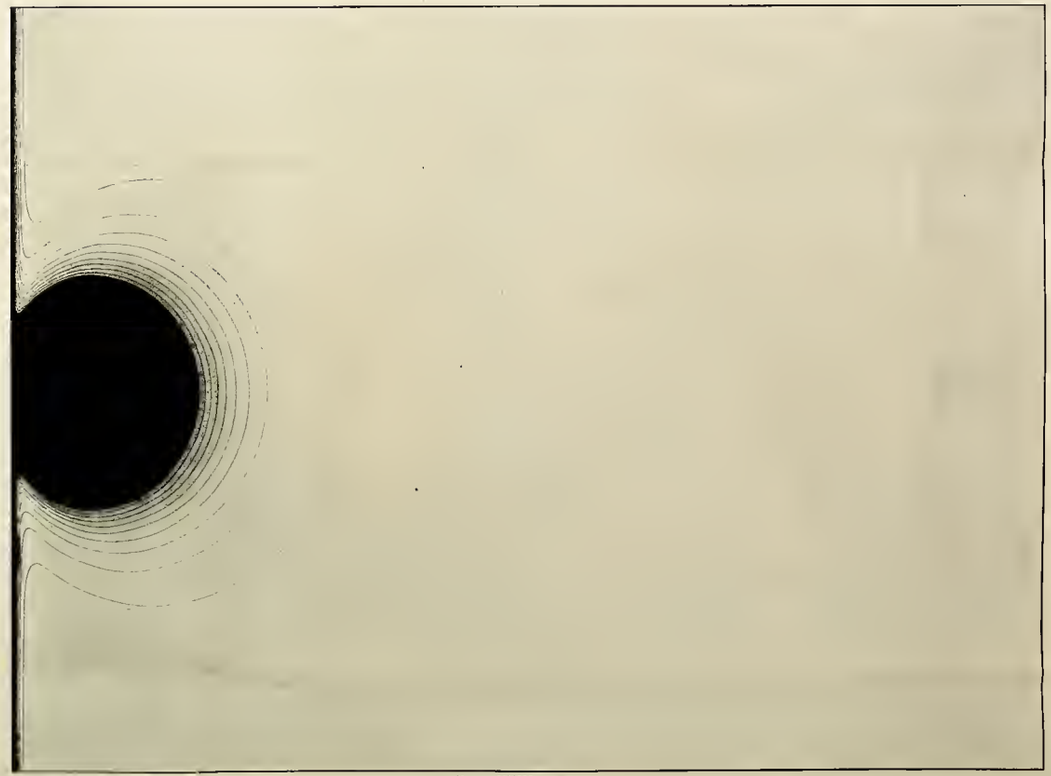


PLATE XVIII $\gamma = 0.01$

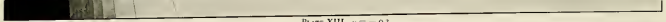


PLATE XIII $\gamma = -0.3$

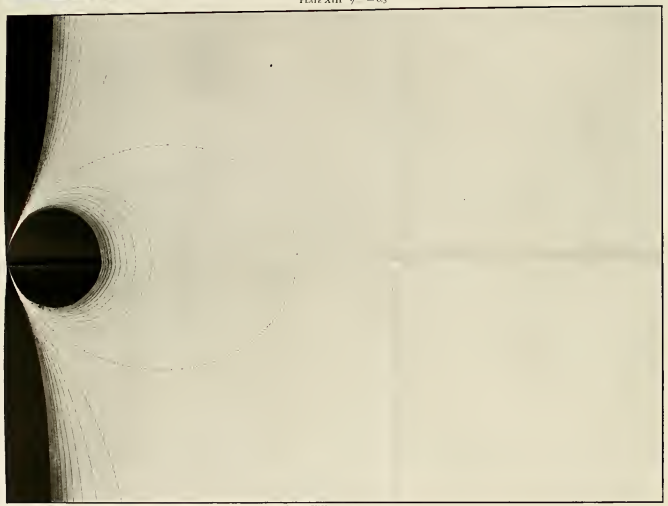


PLATE XIV $\gamma = -0.2$

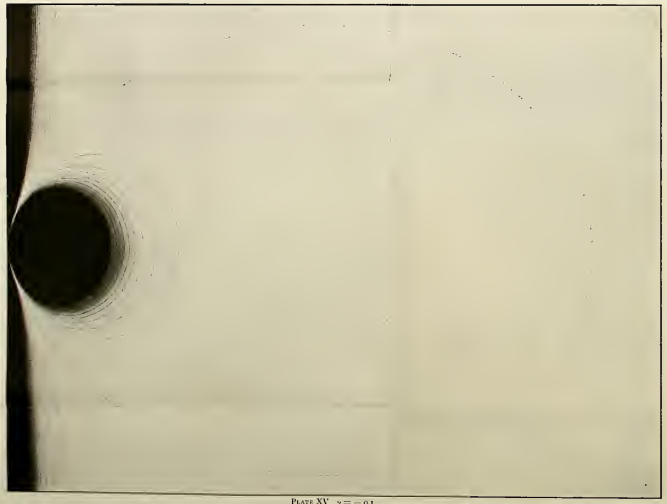
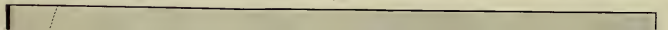


PLATE XV $\gamma = -0.1$



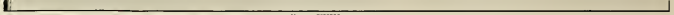


PLATE XVIII $\gamma = 0.01$

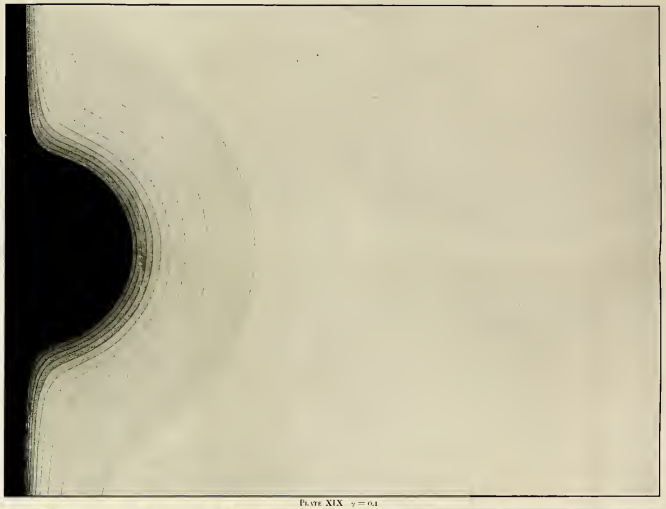


PLATE XIX $\gamma = 0.1$

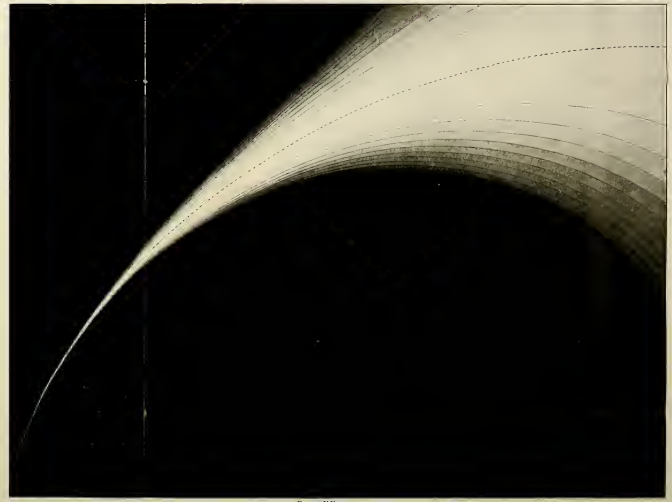


PLATE XX $\gamma = 0.5$

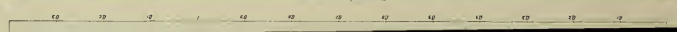


PLATE XV $\gamma = -0.1$

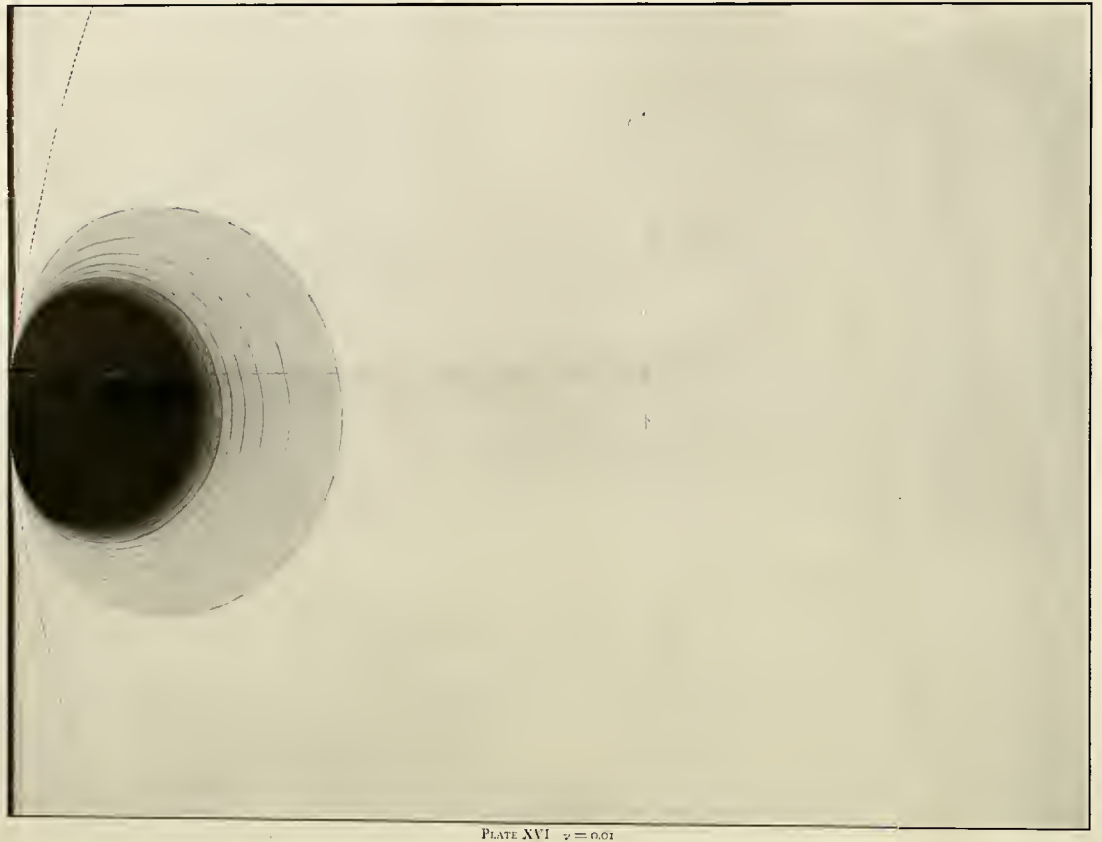


PLATE XVI $\gamma = 0.01$

PLATE XX $\gamma = -0.5$

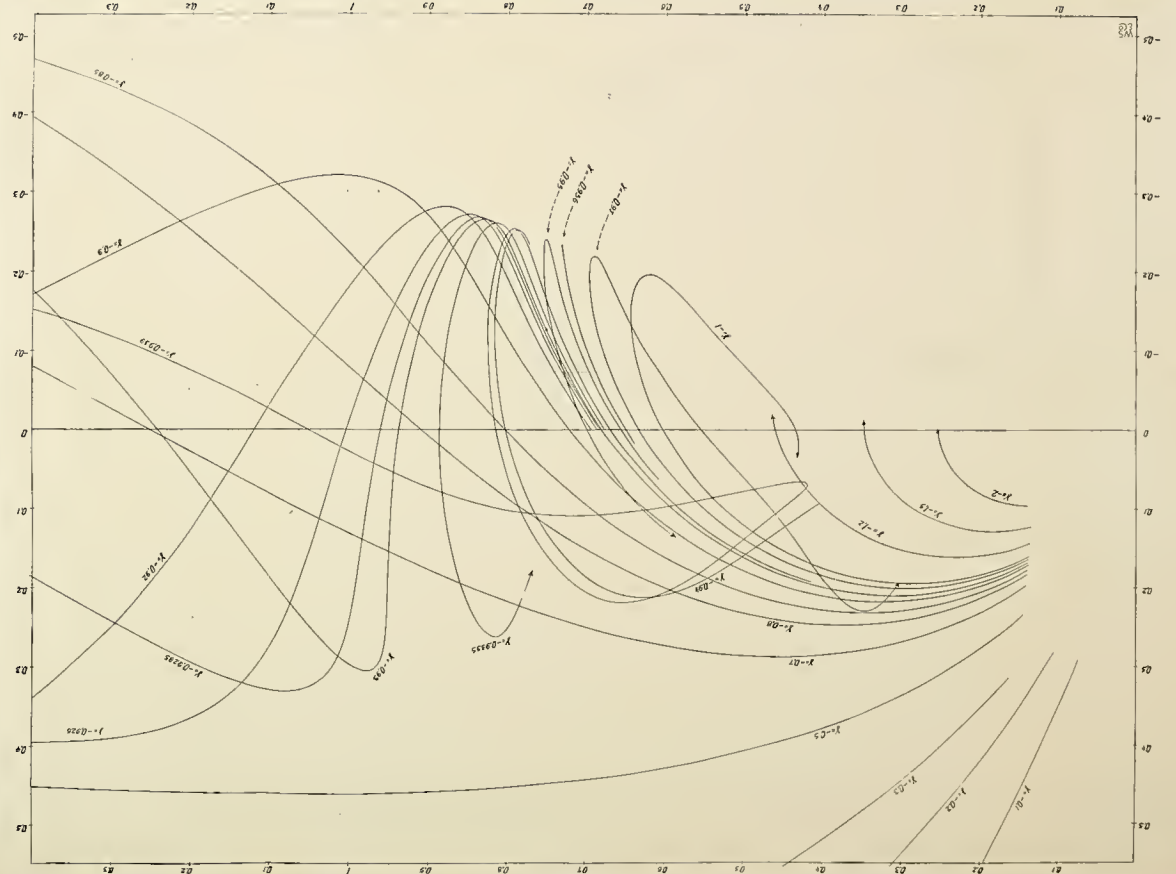


PLATE XXI

PLATE XXIII

PLATE XXIV

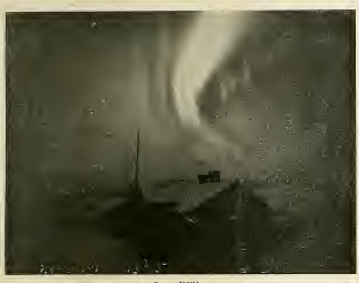


PLATE XXV

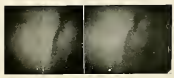


PLATE XXVI*a*

PLATE XXVI*b*

PLATE XXVII*a*

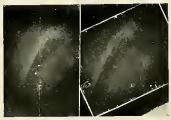


PLATE XXVI*c*

PLATE XXVII*b*

PLATE XXVIII

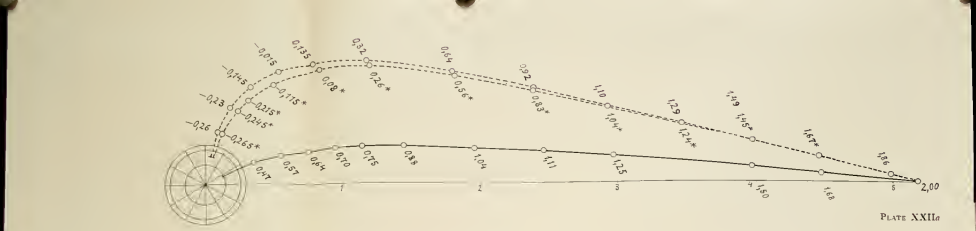


Plate XXIIa

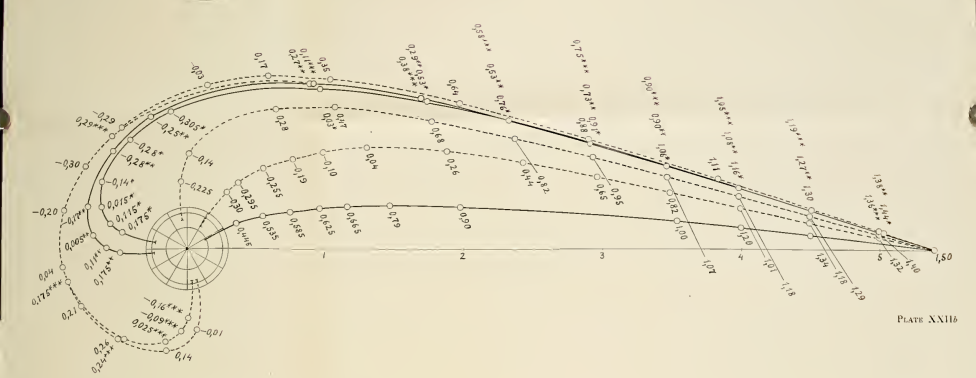


Plate XXIIb

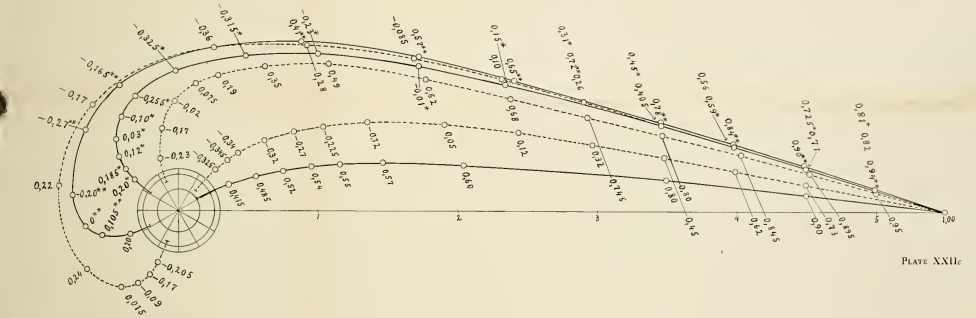


Plate XXIIc

Plate XXIId

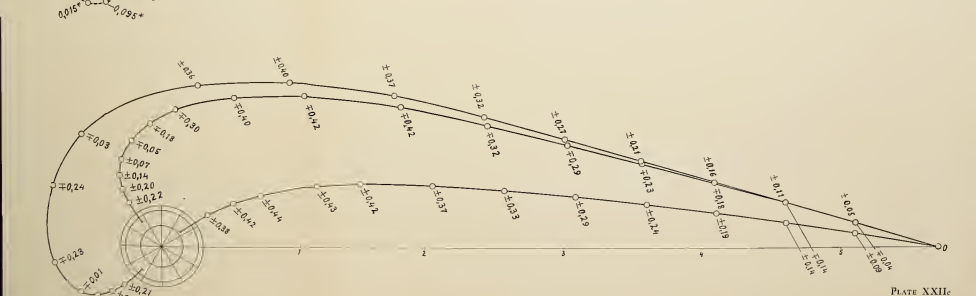


Plate XXIIe

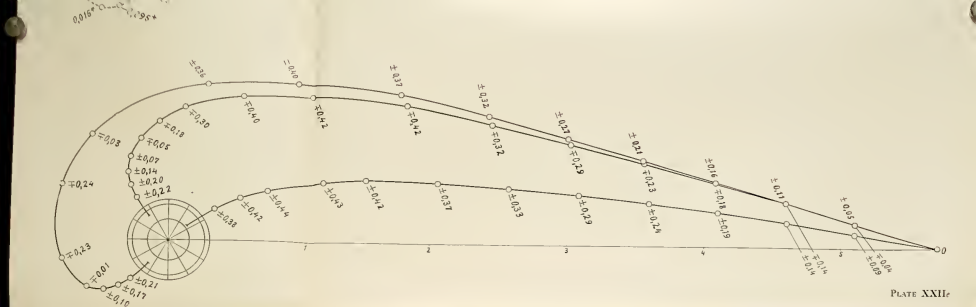


Plate XXII*e*

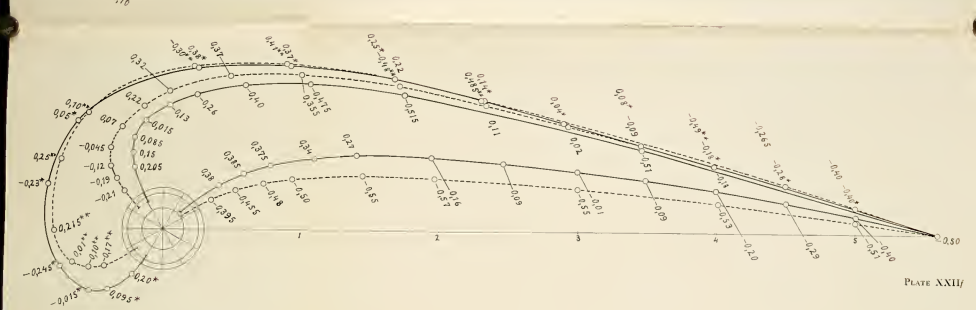


Plate XXII*f*

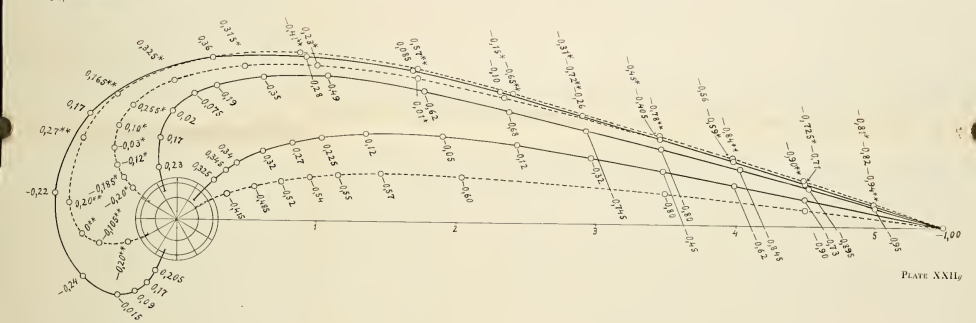


Plate XXII*g*

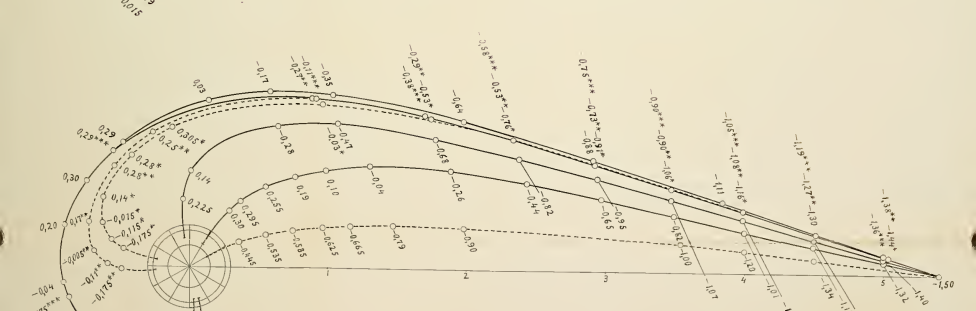


Plate XXII*h*

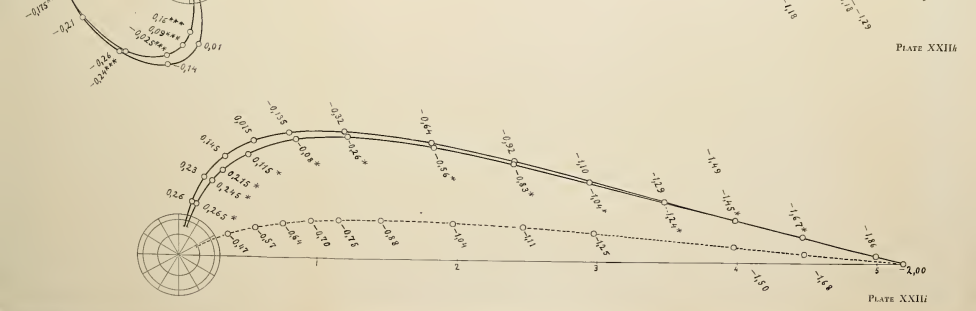


Plate XXII*i*

FIGURE 1

FIGURE 2

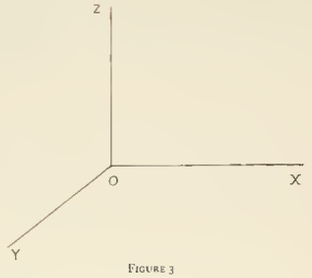


FIGURE 3

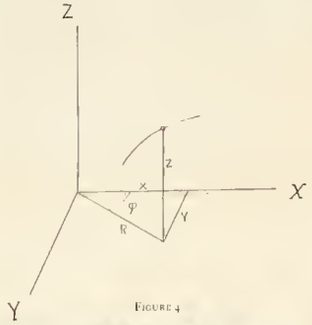


FIGURE 4

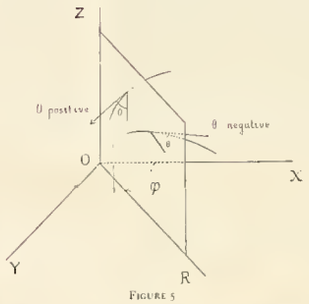


FIGURE 5

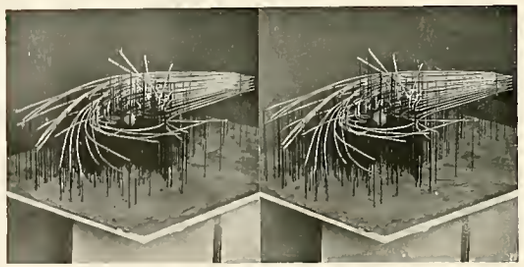


FIGURE 6

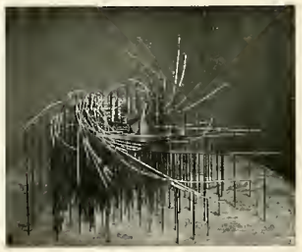


FIGURE 7

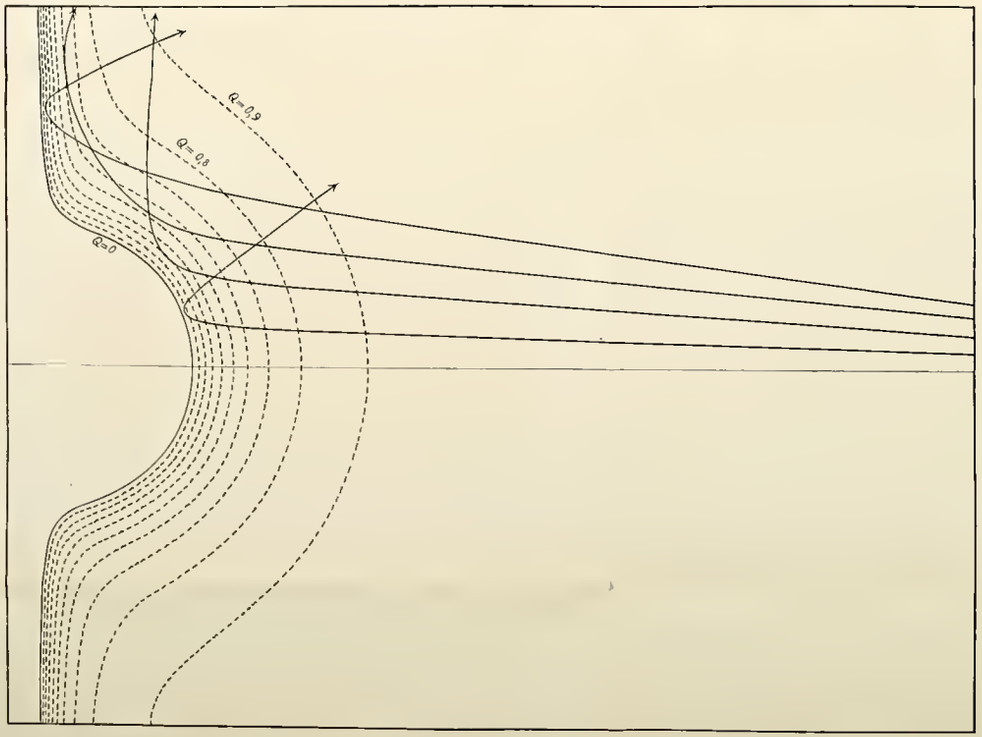


FIGURE 8

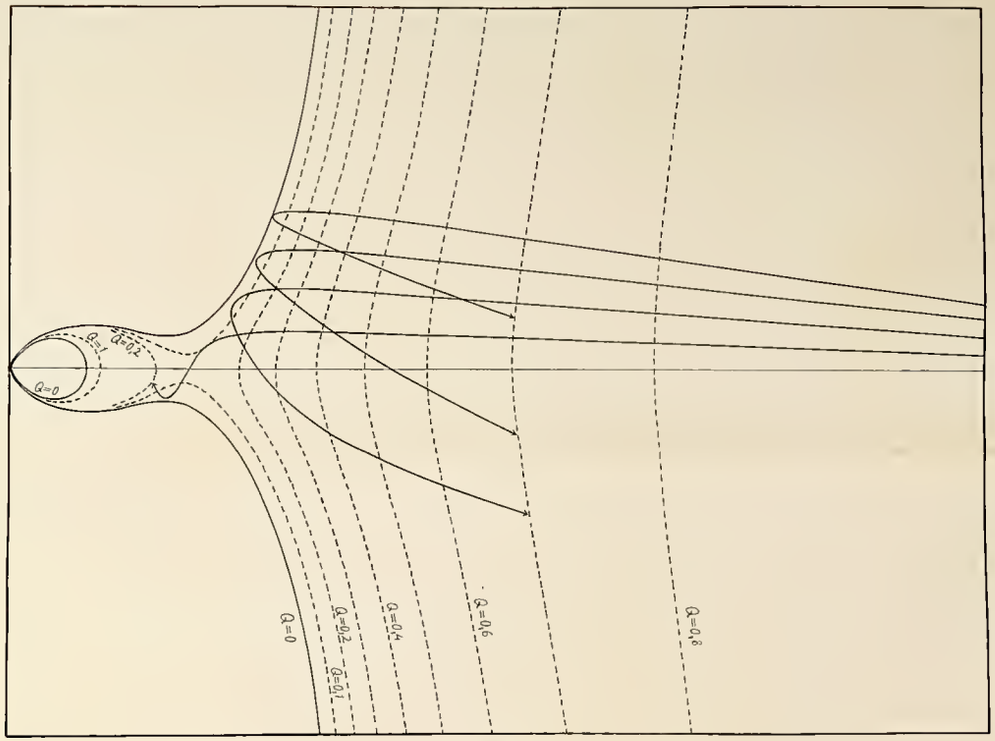


FIGURE 11

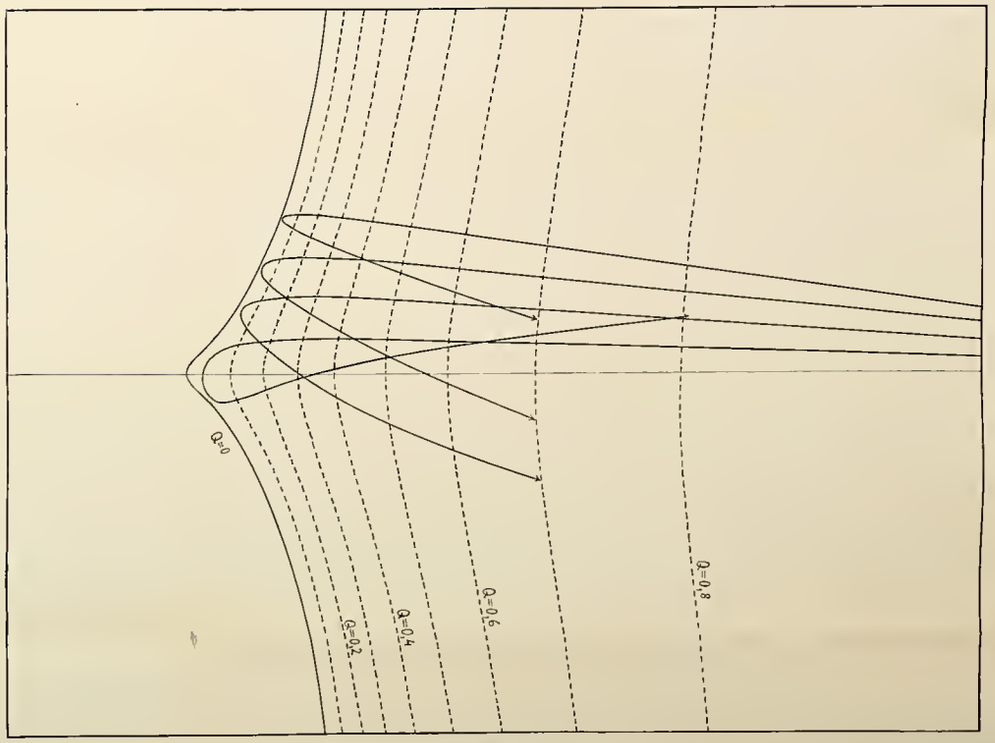


FIGURE 12

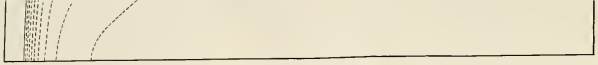
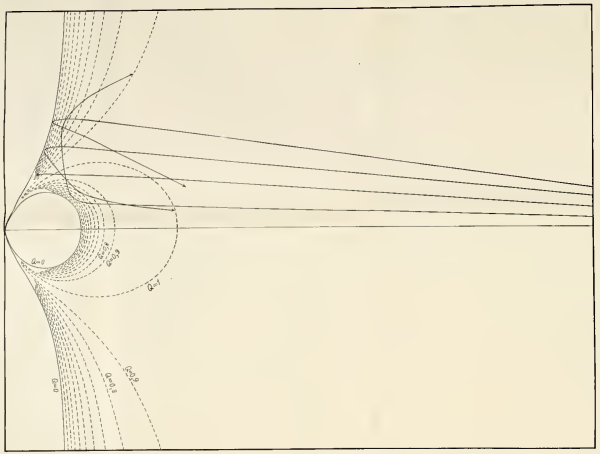


FIGURE 8



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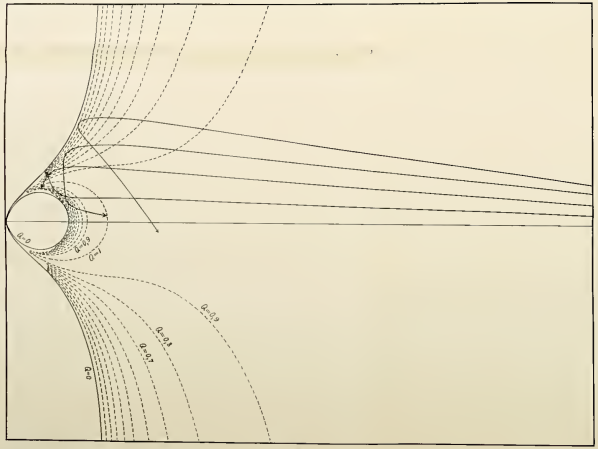


FIGURE 10

FIGURE 12

FIGURE 13

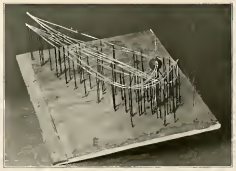
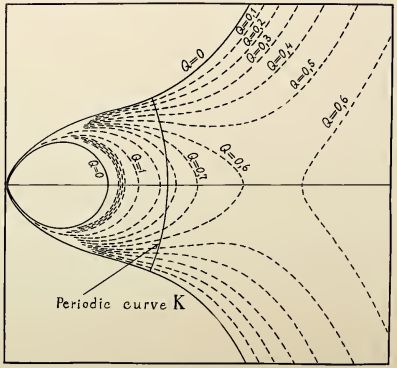


FIGURE 14



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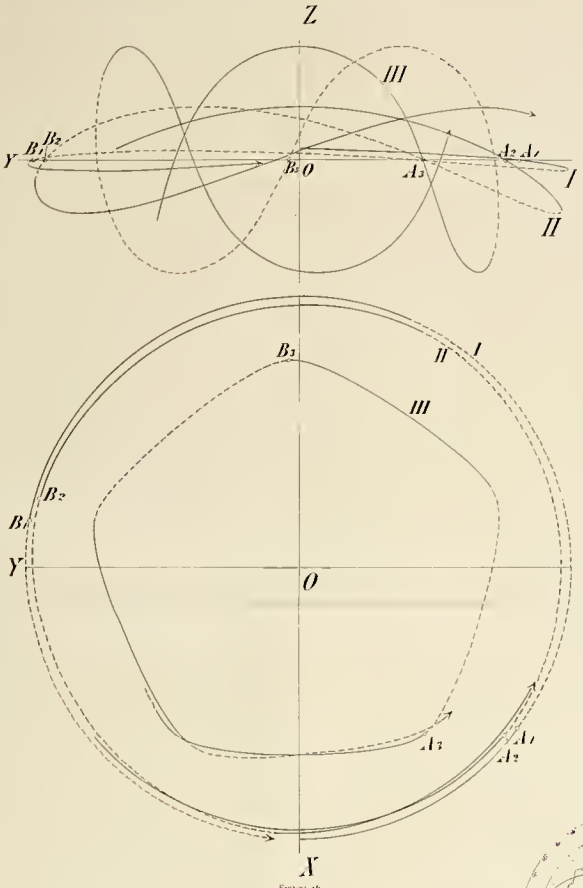


FIGURE 16

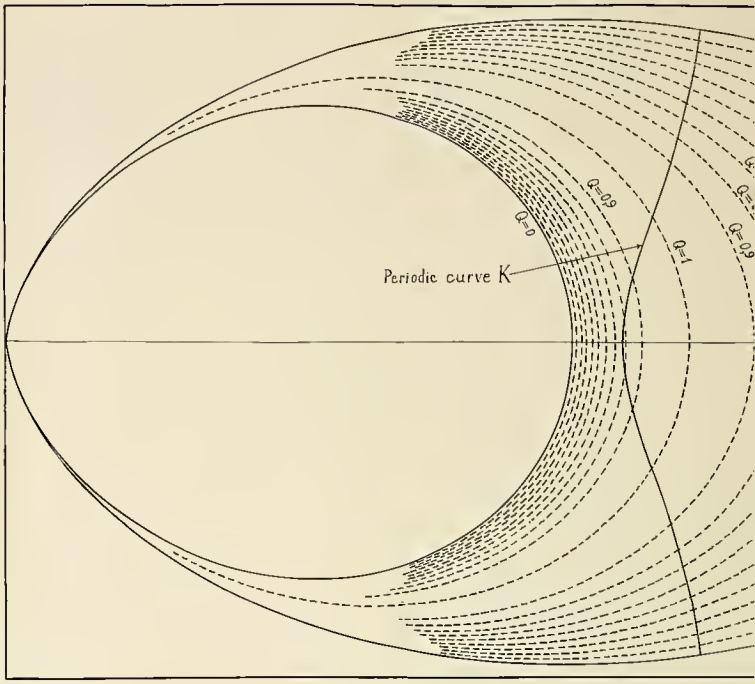


FIGURE 17

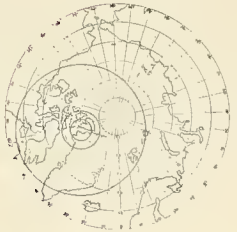


FIGURE 23a

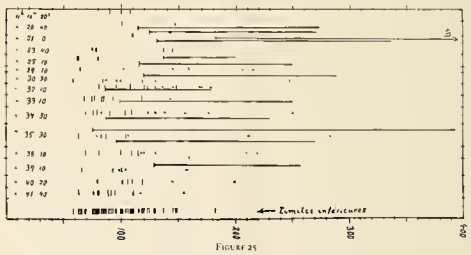


FIGURE 25

FIGURE 22

FIGURE 23b

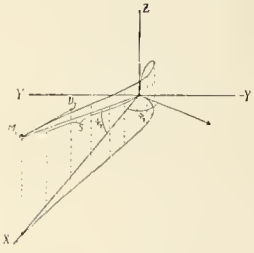


FIGURE 26

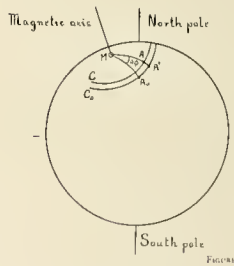


FIGURE 27

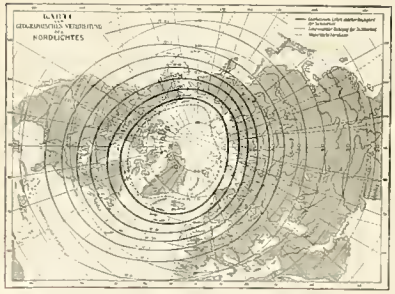
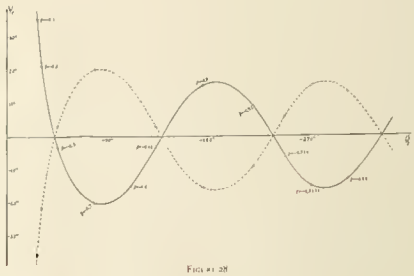


FIGURE 24



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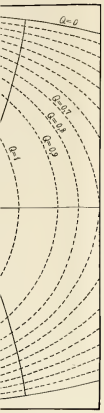
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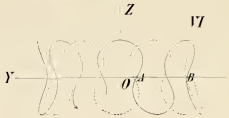
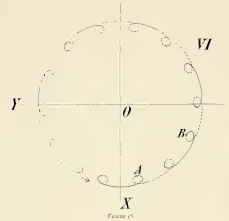

Z
VI
Y
O
A
B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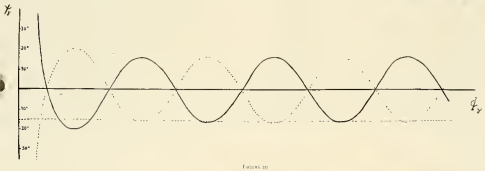

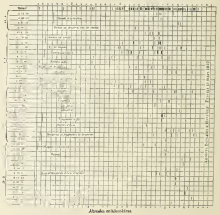

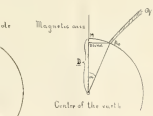
Magnetic axis
Centre of the earth